Lecture 9 Fusion and modular cats II – modularity

In this talk $\mathcal{C}$ will be a ribbon fusion category. Recall that this means that

- We have a set of simples $\{S_1, \dots, S_r\}$
- We have a braiding $T_{ij}: S_i S_j \to S_j S_i$
- We have a pivotal structure $S_i, S_i^*, S_i^{**} \simeq S_i$
- $\mathcal{C}$ is spherical and thus $\dim^{\ell}_{\mathcal{C}}(S_i) = \dim^{r}_{\mathcal{C}}(S_i)$ $= \dim_{\mathcal{C}}(S_i)$

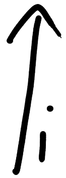
Def 9.1 The dimension of $\mathcal{C}$ is defined to be

independent of $\{S_i\}$

spherical

$$\dim \mathcal{C} = \sum_i \dim^{\ell}_{\mathcal{C}}(S_i) \dim^{r}_{\mathcal{C}}(S_i) \overset{\text{spherical}}{=} \sum_i \dim_{\mathcal{C}}(S_i)^2$$

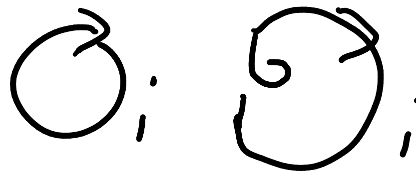
Lemma 9.2 For all i there is a scalar $v_i \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ such that
non-zero
$\cong \mathbb{K}$

 $= v_i$ 

Proof By Schur, $\mathrm{Hom}_{\mathcal{C}}(S_i, S_i) = \mathbb{K}\{\mathrm{id}_{S_i}\}$

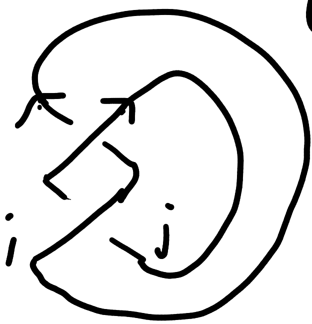
Lemma 9.3 We have

$\Delta_+ := \sum_i$  $= \sum_i v_i^{1} \dim(S_i)^2$

$\Delta_- := \sum_i$  $= \sum_i v_i^{-1} \dim(S_i)^2$

Proof: Clear

For i,j we let

$S_{ij} = tr(T_{ji} \cdot T_{ij}) =$  $\in \mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{K}$

= The value of the i,j – Hopf link

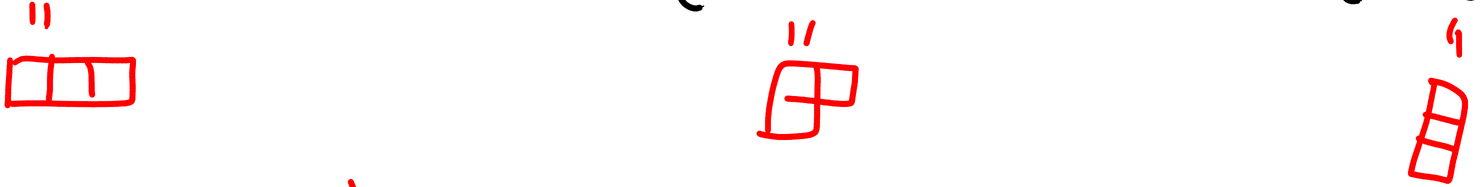
Def 9.4 $\mathcal{C}$ is said to be **modular** if
↑ fusion + ribbon

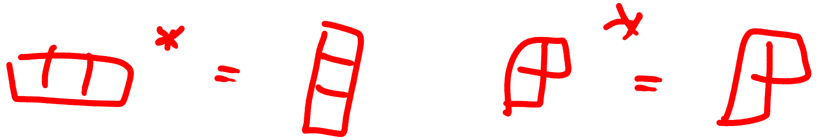
$$S = (S_{ij})_{ij} \in \mathrm{Mat}_{r\times r}(\mathbb{K}) \text{ is invertible}$$

↑ The S-matrix

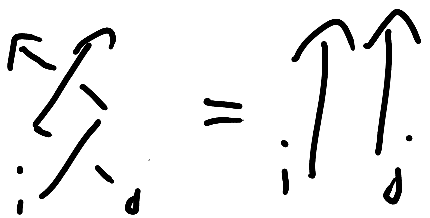
Example Let $\mathcal{C} = \mathbb{C}[S_3]$-mod with symmetric pairing and usual pivotal structure. In particular,

$$\dim_e(S_0) = 1, \quad \dim_e(S_1) = 2 \quad \dim_e(S_2) = 1$$





and



$$\Rightarrow \quad S_{ij} = \dim_e(S_i)\dim_e(S_j)$$

Thus, we get the 3×3 matrix

$$S = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

which has $\det(S) = 0$. So $\mathcal{C}$ is not modular.

Proposition 9.5 If $\mathcal{C}$ is modular, then $\dim \mathcal{C} = \Delta_+ \Delta_-$ and $\underset{\neq 0}{\Delta_+}, \underset{\neq 0}{\Delta_-} \in \mathbb{K}^*$ ($\Rightarrow \dim \mathcal{C} \neq 0$)

Proof A slightly annoying calculation shows that $S^2 = \dim \mathcal{C} \cdot \mathrm{id}_{r\times r} = \Delta_+ \Delta_- \mathrm{id}_{r\times r}$

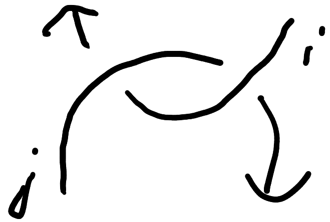
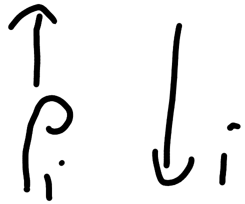
This implies the claim.

Proposition 9.6 We have the Verlinde formula S

$$\sum_i \frac{S_{ij}\, S_{ik}\, S_{ie^*}}{\dim(S_i)} = \dim(e)\, N_{jk}^{e}$$

The fusion rules determine S

Proof Omitted.

Define $\tilde S_{ij} = \dfrac{\dim(S_i)}{\sqrt{\dim e}}$  $\tilde T_{ij} = \delta_{ij}$ 

$\tilde C_{ij} = \delta_{ij^*}$  $\rightsquigarrow$ Matrices $\tilde S = (\tilde S_{ij})$ etc.

Proposition 9.10 – $\tilde S^2 = \tilde C$

– $(ST)^3 = \sqrt{\Delta_+/\Delta_-} \cdot S^2$

Proof: Omitted

One gets even more numerical properties of S modular $\mathcal{C}$. One can even classify them.
However, let us address the crucial question:

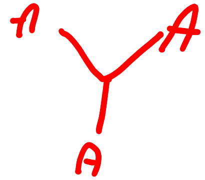
How to construct modular cats?

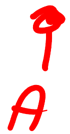
Example If $\mathcal{C}$ is symmetric, then $T_{ij}T_{ij} = 1$ and thus, $S = (\dim_{\mathcal{C}}(S_i)\dim_{\mathcal{C}}(S_j))$
This implies that S is degenerate unless $r = 1$.

Thus, $\mathrm{Vect}_{\mathbb{K}} = \mathrm{Vect}(1)$ is the only modular cat among $\mathrm{Vect}(G)$ (with the usual structure)

Recall that we have seen bialgebras:

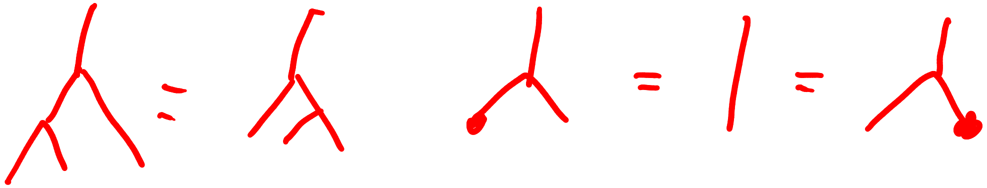
mult 

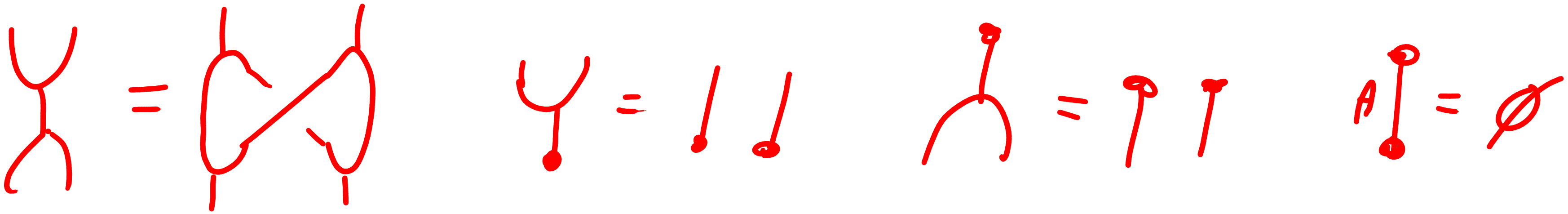
comult 

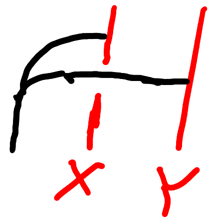
unit 

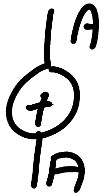
counit 

and that


+ conversions

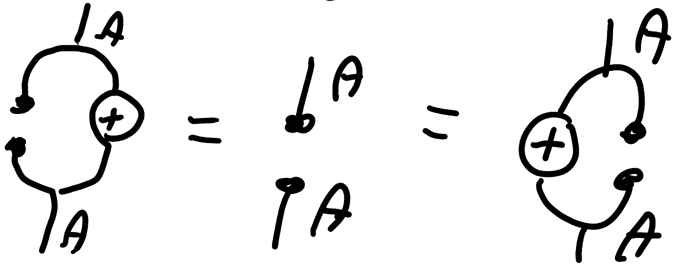


Recall that Amod is monoidal via 

Def 9.11 An *antipode* S of $A \in \mathcal{C}$ is a morphism 

and that

bialgebra

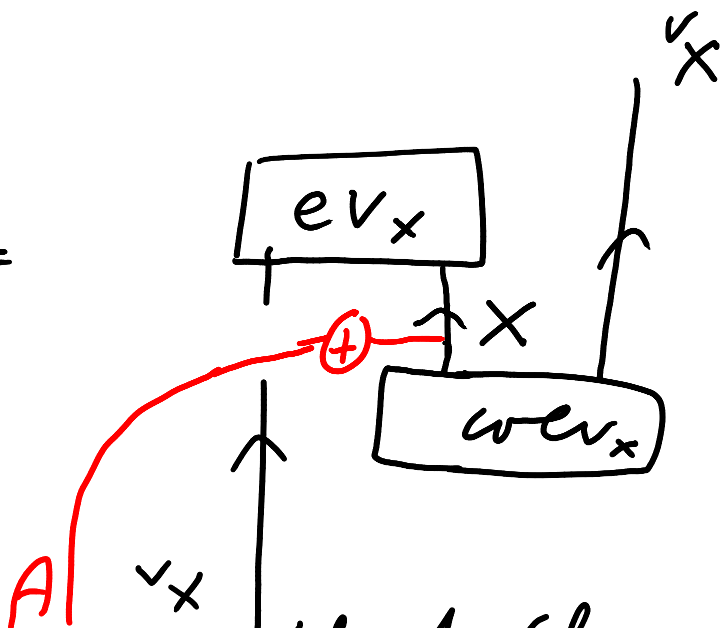


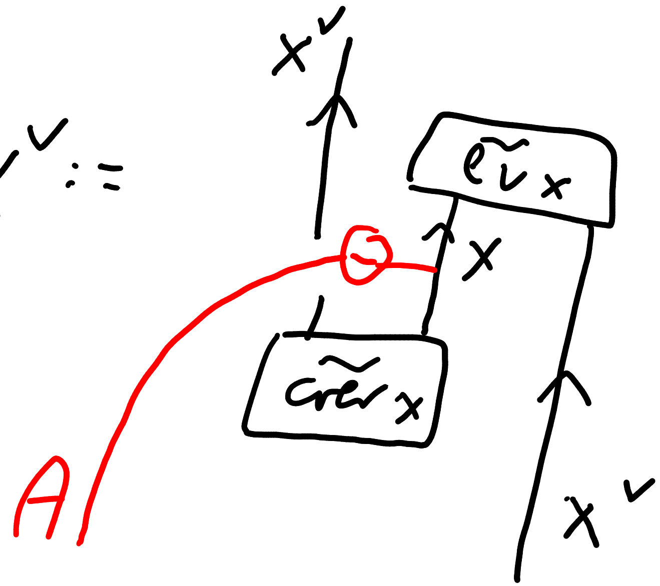
⟹ if invertible depict the inverse as 

Def 9.12 A Hopf algebra $H \in \mathcal{C}$ (braided + rigid) is a bialgebra with an invertible antipode

Theorem 9.13 Hmod of H-modules in $\mathcal{C}$ is a rigid category

Proof Define for $X \in A$mod a A-action on ${}^\vee X, X^\vee$:

${}^\vee X :=$ 

$X^\vee :=$ 

One checks that these are left/right duals.

For a finite group G we have Hopf alg. in $\mathrm{Vect}_{\mathbb{K}}$:

	$\mathbb{K}[G]$	$\mathcal{F}(G) = \mathrm{Hom}(G, \mathbb{K})$
basis	$\{g\}$	$\{g^*\}$, $g^*(h) = \begin{cases} 1, & g=h \\ 0, & g\neq h \end{cases}$
mult	$m(g\,h) = gh$	$m(g^* h^*) = \delta_{gh}\, g^*$
comult	$\Delta(g) = g\, g$	$\Delta(g^*) = \sum_{jk=g} j^* k^*$
unit	$\iota(1) = 1$	$\iota(1) = \sum g^*$
counit	$\varepsilon(g) = 1$	$\varepsilon(g^*) = g^*(1)$
antipode	$S(g) = g^{-1}$	$S(g^*) = (g^{-1})^*$

These are $\cong$ as vector spaces and they will be the main ingredients of the quantum double of G

Def 9.14 The **quantum double $D(G)$** of G is defined to be the Hopf algebra in $\mathrm{Vect}_{\mathbb{k}}$ with

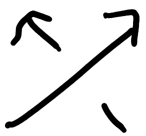
basis $\{g^* h\}$ mult $m(g^* h \; i^* j) = \delta_{g^* h, h i^*} \cdot g^* h j$

comult $\Delta(g^* h) = \sum_{ij = g} (i^* h)(j^* h)$

unit $\iota(1) = \sum_g g^*$ counit $\varepsilon(g^* h) = \delta_{g,1}$

antipode $S(g^* h) = (h^{-1} g^{-1} h)^* h^{-1}$ Actually DGmod is pivotal since $S^2 = 1$

Lemma 9.15 Let $R = \sum_g (g^*) g$ and Θ be the swap of tensor factors.

Then  $= \Theta R$ gives $D(G)$mod the structure of a braided, pivotal, ribbon cat.

Proof: We will discuss this next time.

Note that both, $\mathbb{K}[G]$ as well as $\mathcal{F}(G)$ are subalgebras of $D(G)$ ← dim is $|G|^2$

$\Rightarrow$ Each $D(G)$-module M is a G-graded VS

$M = \bigoplus_{g\in G} M_g$ such that $h . M_g \subset M_{hgh^{-1}}$

and actually any $X \in D(G)\text{mod}$ is of this form. We swap to $\mathbb{K} = \mathbb{C}$.

Let $Z(g)$ be the centralizer of g, and let π be any simple of $Z(g)$.

Define $V_{g,\pi} = \mathbb{C}\{xv \mid x \in G, xgx^{-1} \sim_{conj} g, v \in \pi\}$ and action

$$f^* h . xv = \delta_{f, hxgh^{-1}x^{-1}} \cdot hxv$$

Lemma 9.16 $V_{g,\pi}$ are simple $D(G)$-modules 12 and $V_{g,\pi} \simeq V_{g',\pi'}$ iff $\underbrace{g \sim_{\text{conj}} g'}_{\Rightarrow Z(g) = Z(g')}$ and $\pi \simeq \pi'$

Proof. Omitted

Example For $G = \mathbb{Z}/2\mathbb{Z} = \langle 1, \sigma \rangle$ we have 2 conj classes $\{1\}$, $\{\sigma\}$ and $Z(1) = Z(g) = G$ has 2 simple $S_1 = \text{triv}$, $S_2 = \text{sgn}$.

	Action	$1^* 1$	$1^* \sigma$	$\sigma^* 1$	$\sigma^* \sigma$
$V_{1,S_1} = \mathbb{C}1$		1	1	0	0
$V_{1,S_2} = \mathbb{C}1$		1	-1	0	0
$V_{\sigma,S_1} = \mathbb{C}1$		0	0	1	1
$V_{\sigma,S_2} = \mathbb{C}1$		0	0	1	-1

Lemma 9.17 $D(G)$ mod is a ribbon, fusion cat with simple objects $\{V_{\bar{g},\pi}\}$

Proof The given formulas and arguments above show that $D(G)$ mod is ribbon.
The only non-trivial fact is that it is semisimple.
To see this recall that $\mathcal{C} = A$mod semisimple iff

$$\sum (\dim S_i)^2 = \dim A$$ ← Good trick to remember

We now simply compute, using that $Z(G)$ is semisimple and the orbit formula:

$$\sum_{\bar{g},\pi} (\dim V_{\bar{g},\pi})^2 = \sum_{\bar{g},\pi} |\bar{g}|^2 \dim \pi^2 = \sum_{\bar{g}} |\bar{g}|^2 |Z(G)| = |G|^2$$

Theorem 9.18 $D(G)\mathrm{mod}$ is modular with simple objects $\{V_{\bar{g},\pi}\}$ and

- $(V_{\bar{g},\pi})^* \simeq V_{\overline{g^{-1}},\pi^*}$
- $$S_{\bar{g},\pi,\bar{g}',\pi'} = \frac{1}{|Z(g)||Z(g')|} \sum_{\substack{h\in G \\ hg'h^{-1}\in Z(g)}} \mathrm{tr}_\pi(h g'^{-1} h^{-1})\, \mathrm{tr}_{\pi'}(h^{-1} g^{-1} h)$$
- $$T_{\bar{g},\pi,\bar{g}',\pi'} = \delta_{g,g'}\,\delta_{\pi,\pi'} \frac{\mathrm{tr}_\pi(g)}{\dim \pi}$$

Proof Uses character theory of finite groups and is omitted.

Example Back to $G=\mathbb{Z}/2\mathbb{Z}$. We had four simples $V_{1,\square}$, $V_{1,\boxminus}$, $V_{\sigma,\square}$, $V_{\sigma,\boxminus}$

The T-matrix is

$$T=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

order $V_{1,\square}, V_{1,\boxminus}, V_{\sigma,\square}, V_{\sigma,\boxminus}$

The S-matrix is

$$S=\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$S^2=id$

$(ST)^3=id$

The modular group is, by definition,

$SL_2(\mathbb{Z})$ or $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/_{\pm 1}$

$\simeq$ Möbius transfos

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d} \qquad z \in \mathbb{C} \cup \{\infty\}$$

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Clearly $S, T \in SL_2(\mathbb{Z})$ or $S, T \in PSL_2(\mathbb{Z})$

<u>Lemma 9.19</u> $SL_2(\mathbb{Z}) \simeq \langle S, T \mid (ST)^3 = S^2, S^4 = 1 \rangle$

$PSL_2(\mathbb{Z}) \simeq \langle S, T \mid (ST)^3 = S^2 = 1 \rangle$

<u>Proof</u> omitted

Theorem 9.20 If $\mathcal{C}$ is modular, then

$$S \longmapsto \frac{1}{\sqrt{\dim \mathcal{C}}}\, S \quad , \quad T \longmapsto T$$

define a projective representation of $SL_2(\mathbb{Z})$.

a representation up to scalars

Example For $D(G)$ mod we actually had $S^2 = id$, $(ST)^3 = id$ $\rightsquigarrow$ action of $PSL_2(\mathbb{Z})$