Lecture 10 Quantum invariants – finally

Recall 1 State $= \mathrm{fTan}^{or}$ = "oriented, framed tangles"

This is the free braided pivotal cat gen by one object $+ (\to, \uparrow)$ and

$$\mathrm{fTan}^{or} = \langle +, \times, \curvearrowright, \curvearrowleft, \cup, \cup \mid \text{or. Reidemeister moves} \rangle$$

Corollary 10.1 For any braided pivotal cat $\mathcal{C}$ there exists a functor

$$\begin{aligned} F: \mathrm{fTan}^{or} &\longrightarrow \mathcal{C} \\ + &\longmapsto X \end{aligned}$$

Def 10.2 Any pair $(\mathcal{C},F)$ as in Cor 10.1 is called a quantum invariant

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Remark 10.3 There are several flavors of what is called a quantum invariant, eg:

- one could ask $\mathcal{C}$ to have more structure, eg being $\mathbb{K}$-linear, $\oplus$ or abelian to do some algebraic calculation (all my examples will be alg. in nature)
- a quantum invariant of a 3manifold is exactly the same but $\mathcal{C}$ being modular

(3manifolds can be cooked up from $\mathcal{T}ang^{\sim}$, but have extra relations)

Example Let $\mathcal{C} = TL = \langle \bullet,\ \cap,\ \cup \mid ∿ = | = ∽,\ \bigcirc = -q-q^{-1} \rangle$

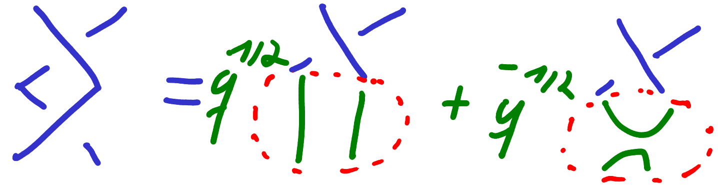
$\mathbb{K}(q^{1/2})$-linear

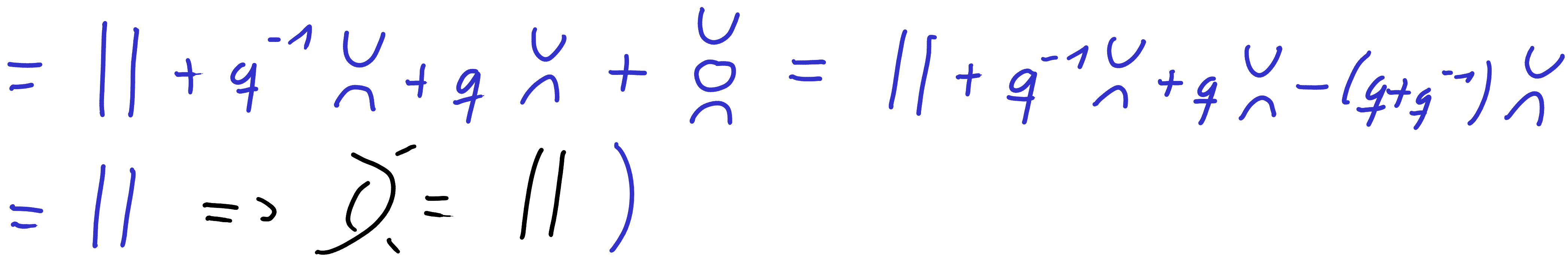
formal variable

With $\times := q^{1/2}\,|| + q^{-1/2}\,\substack{\cup\\\cap}$ one can check that TL is braided pivotal.

$\rightarrow \overline{\times} := q^{-1/2}\,|| + q^{1/2}\,\substack{\cup\\\cap}$

(Example calculation:





)

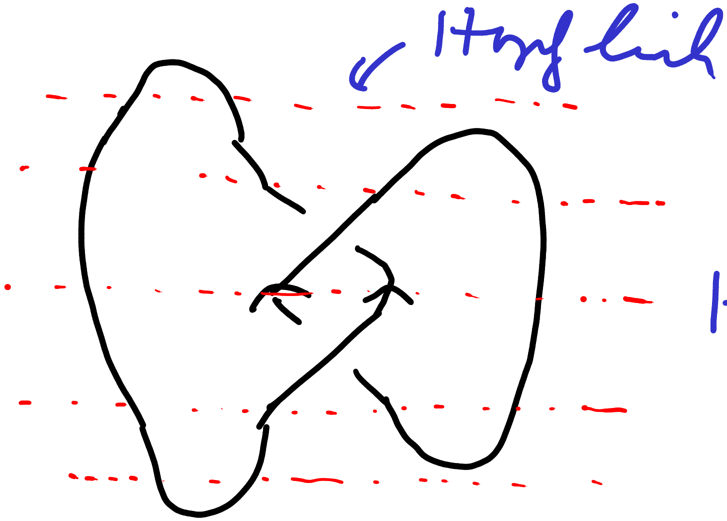
So we get a functor $F: \mathrm{s1Tan}^{or} \longrightarrow TL$

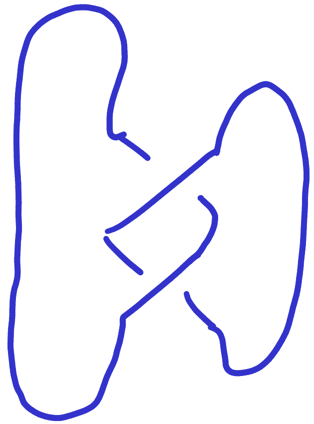
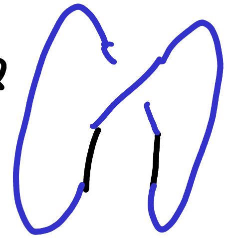
$+ \longmapsto \bullet$

The quantum invariant in this case is the so called Jones polynomial

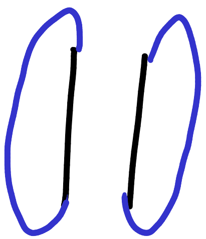
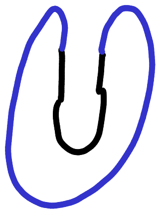
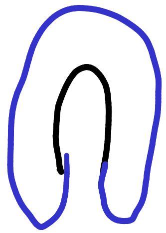
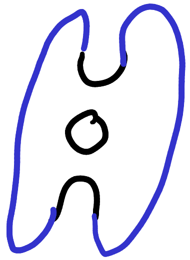
Eg. $f \in \mathrm{End}_{\mathrm{frTan}}(\mathbb{1}) \longmapsto f \in \mathrm{End}_{TL}(\mathbb{1}) = \mathbb{K}(q^{1/2})$

Unlink $\bigcirc \; \bigcirc \longmapsto (q+q^{-1})^2 = q^2 + 2 + q^{-2}$ "a polynomial"



$\longmapsto$  $= q^{1/2}$  $+ q^{-1/2}$ 

Hopf link is $\Leftarrow$ non-trivial

$= q$  $+$  $+$  $+ q^{-1}$  $\underset{\text{Def}}{\overset{\text{Jones}}{=}} q^3 + q + q^{-1} + q^{-3}$

$(q+q^{-1})^2 \qquad -q-q^{-1} \qquad -q-q^{-1} \qquad (q+q^{-1})^2 \qquad\qquad \neq q^2+2+q^{-2}$

Main question: How to construct good braided pivotal categories?

- being able to do calculations
- non-trivial invariants

- TL is a very good example, but it is unclear how this should generalize
- H mod for Hopf algebras H are very good candidates. Catch: The first examples one comes up with, $\mathbb{K}[G]$ or $U(\mathfrak{g})$, give symmetric categories
  - useless for topology
- $D(G)$mods are modular, but give pretty stupid invariants

So: Enter quantum groups

Main idea: quantize the Chevalley presentation of the Lie algebra $\mathfrak{g}$ Here only $\mathfrak{g}=\mathfrak{sl}_2$, but this works very general

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Example (Chevalley)

$U(\mathfrak{sl}_2)$ is the algebra generated by E, F, H modulo

- $HE = EH + 2E$; $HF = FH - 2F$
- $FE = EF - H$

$\Big\}$ PBW $\Rightarrow$ basis $\{E^a F^b H^c \mid a,b,c \in \mathbb{N}\}$

This is a Hopf algebra:

$\Delta(E) = E \otimes 1 + 1 \otimes E$ | $\varepsilon(E) = 0$ | $S(E) = -E$

$\Delta(F) = F \otimes 1 + 1 \otimes F$ | $\varepsilon(F) = 0$ | $S(F) = -F$

$\Delta(H) = H \otimes H$ | $\varepsilon(H) = 1$ | $S(H) = H$

Def 10.4 Quantum sl_2 $U_q(sl_2)$ is the $\mathbb{K}(q)$-algebra $\mathcal{F}$ generated by $E, F, K^{\pm 1}$ modulo

- $KK^{-1} = 1 = K^{-1}K$
- $KE = q^2 EK$; $KF = q^{-2} FK$
- $FE = EF - \dfrac{K - K^{-1}}{q - q^{-1}}$

Lemma 10.5 $\{E^a F^b K^c \mid a, b \in \mathbb{N}, c \in \mathbb{Z}\}$ is a PBW type basis of $U_q(sl_2)$

Lemma 10.6 $U_q(sl_2)$ can be endowed with a Hopf alg structure:

$\Delta(E) = E \otimes 1 + K \otimes E$	$\varepsilon(E) = 0$	$S(E) = -K^{-1}E$
$\Delta(F) = F \otimes K^{-1} + 1 \otimes F$	$\varepsilon(F) = 0$	$S(F) = -FK$
$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$	$\varepsilon(K^{\pm 1}) = 1$	$S(K^{\pm 1}) = K^{\mp 1}$

Proposition 10.7 $U_q(sl_2)$ is a flat deformation of $U(sl_2)$.

Proof (sketch) Like almost everything involving quantum groups, making this rigorous is a pain, but the idea is pretty nice:

Set $K = q^H$, $K^{-1} = q^{-H}$. Then:

$EF - FE = \dfrac{q^H - q^{-H}}{q - q^{-1}} \overset{q=1}{\rightsquigarrow}$ Using L'Hospital (no matter how absurd this sounds) we get

$$EF - FE = \lim_{q\to 1} \frac{q^H - q^{-H}}{q - q^{-1}} = \lim_{q\to 1} \frac{Hq^{H-1} + Hq^{-H-1}}{1 + q^{-2}} = \frac{2H}{2} = H;$$

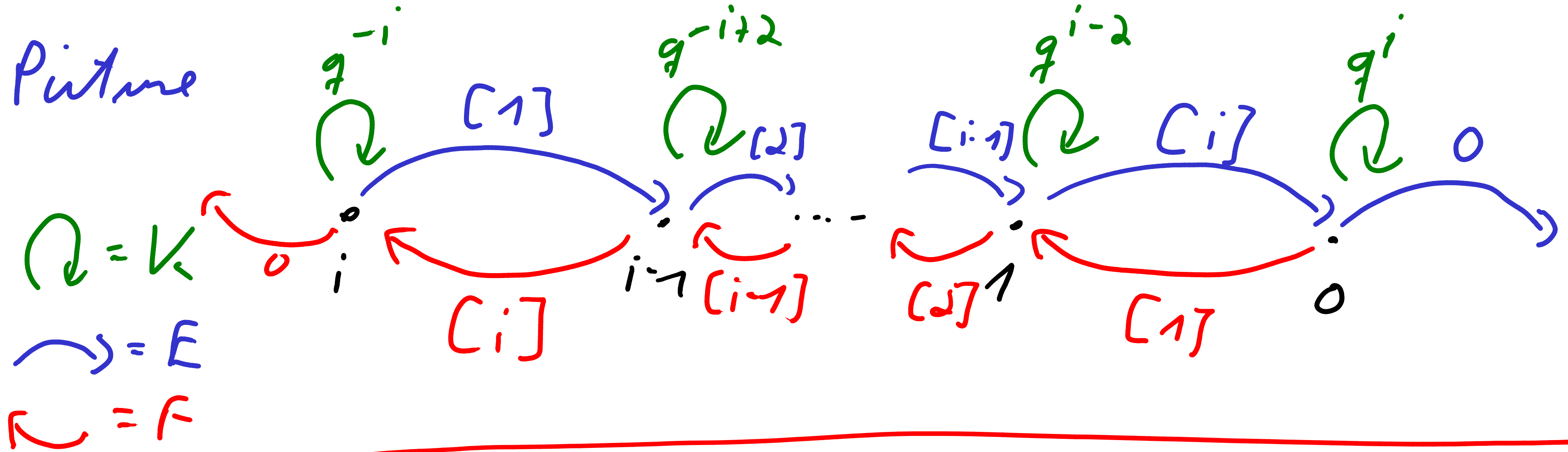
the relation in $U(sl_2)$. The others follow similar.

Let us now study $U_q(\mathfrak{sl}_2)\text{mod}$ $\quad [a] = \frac{q^a - q^{-a}}{q - q^{-1}}$

Def 10.8 For $i \in \mathbb{N}$, the ith Weyl module $\Delta(i)$ is the $\mathbb{K}(q)$-vector space with basis $\{m_0, \ldots, m_i\}$ and action ($m_{<0}, m_{>i}$ are zero, by convention)

$$E m_k = [i-k+1] m_{k-1} \qquad F m_k = [k+1] m_{k+1} \qquad K m_k = q^{i-2k} m_k$$

Picture



One can show that $\dim_{U_q(\mathfrak{sl}_2)\text{mod}} \Delta(i) = [i]$

Theorem 10.9 $U_q(\mathfrak{sl}_2)$mod is an $\mathbb{K}(q)$-linear, $\oplus$, abelian, semisimple cat with simple objects $\{\Delta(i) \mid i \in \mathbb{N}\}$

Moreover, $U_q(\mathfrak{sl}_2)$mod is braided pivotal with

$$\Delta(i)^* \simeq \Delta(i)$$

Finally, for every i $\exists$ $F_i : \mathcal{1}\mathrm{Tan}^{or} \longrightarrow U_q(\mathfrak{sl}_2)$mod, $+ \longmapsto \Delta(i)$
and F_1 recovers the Jones polynomial.

- The above works completely general for $U_q(\mathfrak{g})$
- $U_q(\mathfrak{sl}_2)$mod can be modified into a modular cat
- Rest of this talk: A diagrammatic version of $U_q(\mathfrak{sl}_2)$mod i.e. the TL category

Def 10.10 The category of symmetric $\mathfrak{sl}_2$ webs

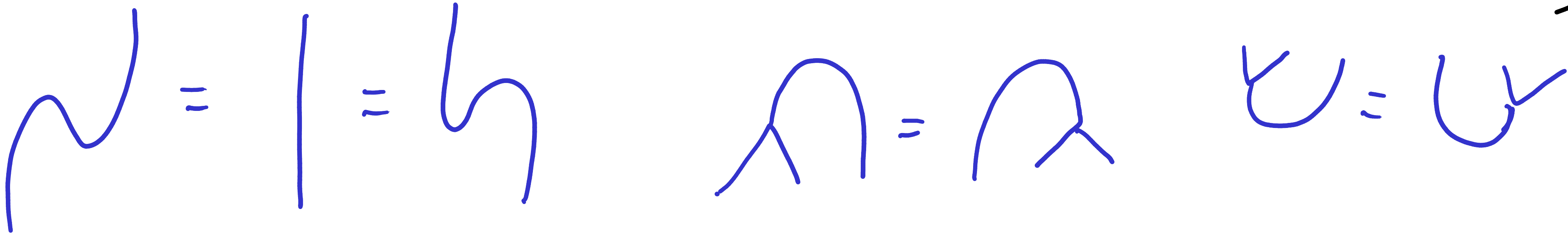
$\mathfrak{sl}_2\mathrm{Web}$ is the $\mathbb{K}(q)$-linear, $\oplus$, monoidal category monoidally generated by

- objects $k \in \mathbb{N}$
- morphisms

"cap" , "cup" , "merge" , "split"

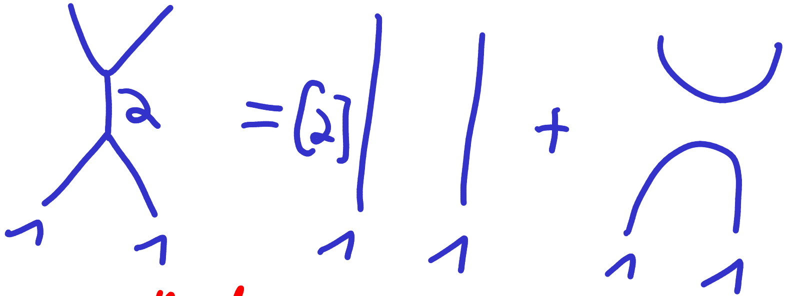
Relations:

"asso", all labels "coasso", all labels "square switch"



"isotopies", all labels

"circle removal"

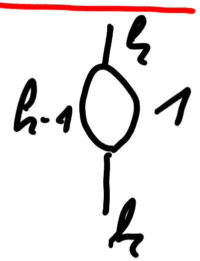


"dumbbell"

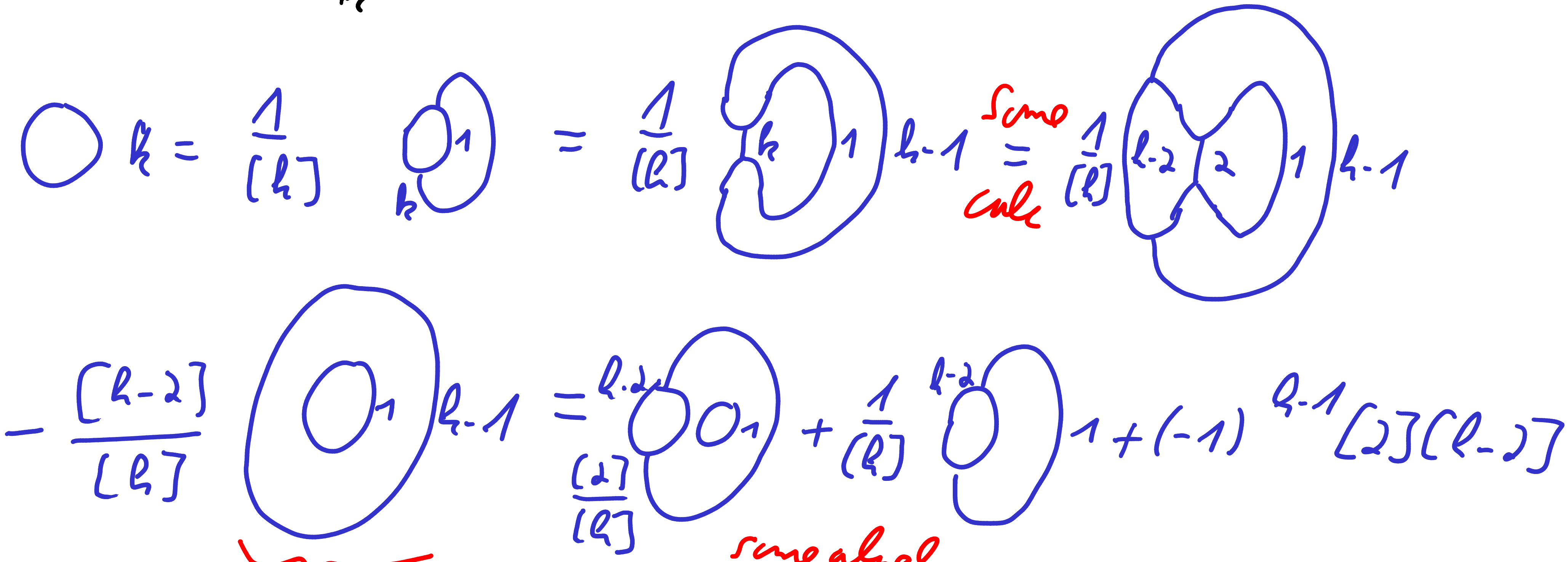
Morphisms in sl_2 Web are thus, labeled trivalent graph up to isotopies and a few combinatorial relations

Lemma 10.11 (Thickening) There are thick versions of the relations, e.g. $\bigcirc_k = (-1)^k [k+1]$

Proof (Sketch) Note that the square switch for $a=k$, $b=0$ becomes



$= [k] \,\Big|_k$



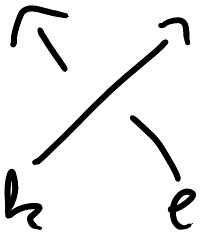
$= (-1)^k [2][k-1]$ by induction

$= \dots = (-1)^k [k+1]$

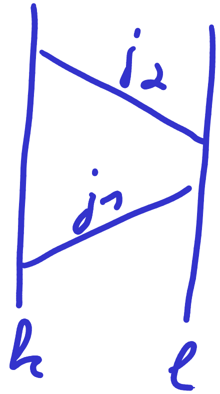
some algebra

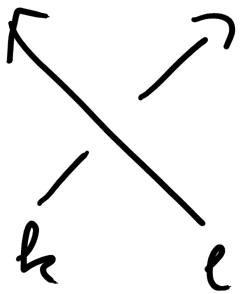
induction

Def 10.12

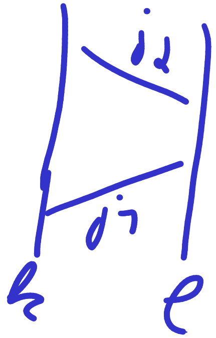


$$:= (-1)^k q^{-k-\frac{kl}{2}} \sum_{j_1 - j_2 = k-l} (-q)^{j_1}$$





$$:= (-1)^k q^{k+\frac{kl}{2}} \sum_{j_1 - j_2 = k-l} (-q)^{-j_1}$$



Lemma 10.13

This endows sl_2 Web with the structure of a braided pivotal category.

Theorem 10.14

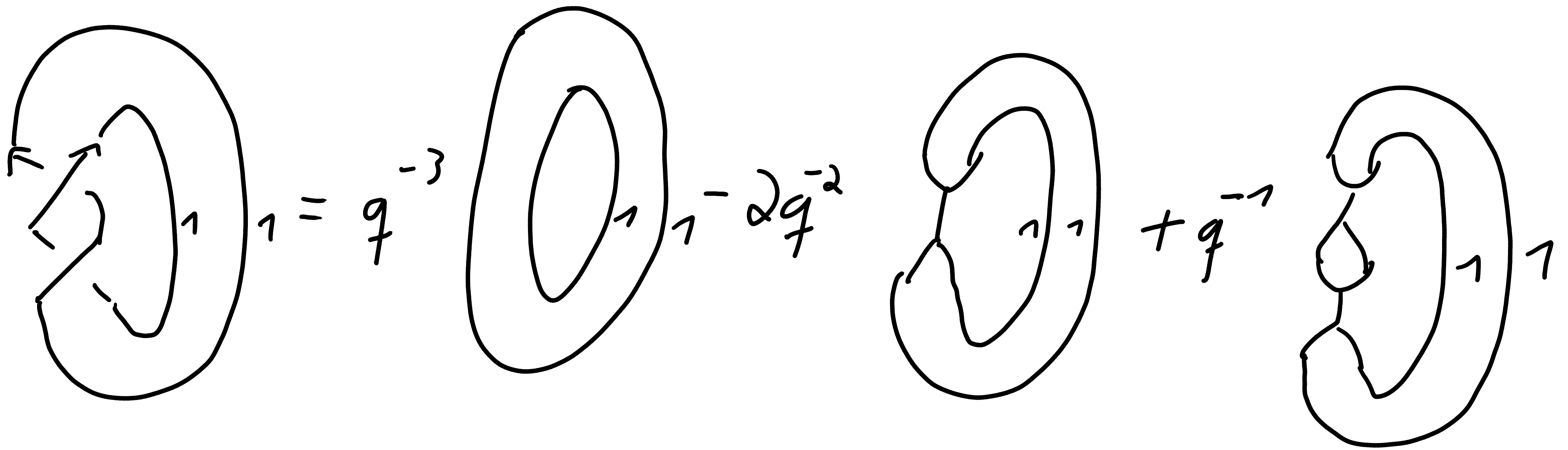
Let Mat($\mathfrak{sl}_2$Web) be the matrix closure of $\mathfrak{sl}_2$Web. Then there exists an equivalence of braided pivotal categories

$$\mathrm{Mat}(\mathfrak{sl}_2\mathrm{Web}) \xrightarrow{\sim} U_q(\mathfrak{sl}_2)\mathrm{mod}$$
$$k \longmapsto \Delta(k)$$

The invariants coming from $\mathfrak{sl}_2$Web are thus, the $U_q(\mathfrak{sl}_2)$ quantum invariants.

Note that they come in families – that is what makes them "quantum"

Example "The colored Jones polynomials of the Hopf link" (Calculation in $\mathfrak{sl}_2$ Web)



$$= q^{-3}[2]^2 - 2q^{-2}[2][3] + q^{-1}[2]^2[3] = [4]$$

So, the Hopf link is non-trivial

In general we have a whole family of these guys and one can show:

$$= (-1)^{k+\ell}\big[(k+1)(\ell+1)\big]$$

$\rightsquigarrow$ Similarly we get whole families for all links.