Emre and Lily

Representation Theory of Algebras
Talk 8: Algebras and Modules I

In this talk we will introduce k-algebras and modules, where k is an algebraically closed field. Since all algebras are also rings, we'll start off by recalling some basics from ring theory.
Then we'll move on to defining k-algebras and give you plenty of examples on how we can connect them to Quivers through path algebras. And lastly we'll present modules and relate them to path algebras and Quivers as well.

4.1 Basics from Ring Theory

Since this section is meant as a refresher on some things that we'll use later, we'll keep it brief and not delve into too many details or give all of the proofs.

Def let R be a ring with $1 \neq 0$.
A <u>right ideal</u> (left ideal) $I \subseteq R$ is a subgroup of the additive group $(R,+)$ s.t. $\forall a \in I, \forall r \in R \quad a \cdot r \in I$ (resp. $r \cdot a \in I$).

If a right ideal is also a left ideal then we'll simply refer to it as an ideal of R, for short.

examples (4.1)

(1) $\{0\} \subseteq R$ and R itself are both ideals of R

(2) Given an ideal $I \subseteq R$ and $m \in \mathbb{N}, m > 0$
$I^m := \{$ all finite sums of elements $a_1 \cdot a_2 \cdot \dots \cdot a_m \mid a_i \in I \}$
is an ideal of R

Def (4.1) An ideal $I \subseteq R$ is called <u>nilpotent</u> if $I^m = 0$ for some $m \in \mathbb{N}, m > 0$.

Def (4.2) A proper (left / right) ideal $I \subset R$ is called maximal if $\forall J \leq R$ ideals s.t. $I \subseteq J \subseteq R$ it holds that either $J = I$ or $J = R$.

Remark: (1) In a commutative ring R an Ideal $I \leq R$ is maximal iff the quotient ring R/I is a field.

(2) If k is a field, then its only ideals are $\{0\}$, k.

Def (4.3) The radical $\operatorname{rad} R$ is the intersection of all maximal right ideals in R

Lemma (4.1) let R be a ring and $a \in R$ then the following statements are equivalent:

(1) $a \in \operatorname{rad} R$
(2) $\forall b \in R$, $1 - ab$ has a right inverse
(3) $\forall b \in R$, $1 - ab$ has a two-sided inverse
(4) a is an element of the intersection of all maximal left ideals
(5) $\forall b \in R$, $1 - ba$ has a left inverse
(6) $\forall b \in R$, $1 - ba$ has two-sided inverse

Corollary (4.2)

(1) $\operatorname{rad} R$ is also equal to the intersection of all maximal left ideals
(2) $\operatorname{rad} R$ is an ideal of R
(3) $\operatorname{rad} (R / \operatorname{rad} R) = 0$
(4) If $I \leq R$ an ideal is nilpotent, then $I \leq \operatorname{rad} R$

4.2. Algebras

In this section we'll define algebras and their properties and give a couple detailed examples, including the path algebra of a quiver.

Def (4.4) let k be an algebraically closed field. A k-algebra A is a ring $(A, +, \cdot)$ with unity 1 s.t. A also has the structure of a k-vector space where:

(1) addition in the vector space A coincides with addition in the ring A

(2) scalar multiplication in the vector space A is compatible with the ring multiplication ie:

$$\forall a, b \in A \quad \forall \lambda \in k, \quad \lambda(a \cdot b) = (\lambda \cdot a) \cdot b = a(\lambda \cdot b) = (a \cdot b) \cdot \lambda$$

Remark the dimension of the algebra A is the dimension of the k-vector space A.

examples (4.2)

(1) The ring of polynomials $k[x]$ in x, is a k-algebra. Its unity is the constant polynomial 1.
Scalar multiplication by $\lambda \in k$ is done by multiplying each coefficient of a polynomial by λ.

We can easily see that multiplication and scalar multiplication in $k[x]$ are compatible:

$f(x), g(x) \in k[x]$, $\lambda \in k$ then:
$$\lambda(f(x) \cdot g(x)) = (\lambda \cdot f(x)) \cdot g(x) = f(x) \cdot (\lambda g(x)) = (f(x) \cdot g(x))\lambda$$

(2) The set of all $n \times n$ Matrices over k, $\mathrm{Mat}_{n\times n}(k)$, is a k-algebra, its unity being the identity matrix.

It should be clear how $Mat_{n\times n}(k)$ is a ring and has a k-vector space structure. Scalar multiplication is just the same as usual.

Again it is easy to check multiplication comptatibility:

$A, B \in Mat_{n\times n}(k)$, $\lambda \in k$ then:

$$\lambda \cdot (A \cdot B) = (\lambda A) \cdot B = A \cdot (\lambda B) = (A \cdot B) \cdot \lambda$$

(3) We look at the set of lower (or upper) triangular matrices. These clearly form a subring of all matrices, and therefore also have a k-Algebra structure

the unity, multiplication and scalar multiplication are the same as in $Mat_{n\times n}(k)$

(4) The set of all 3×3 matrices of the form: $\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix}$

also form a k-algebra.
The identity matrix is in the set.
Let us just check the multiplication:

$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & e \end{pmatrix}$ $B = \begin{pmatrix} v & 0 & 0 \\ 0 & w & 0 \\ x & y & z \end{pmatrix}$ $a, b, c, d, e, v, w, x, y, z \in k$ and $\lambda \in k$

$$\lambda(A \cdot B) = \lambda \cdot \left[\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & e \end{pmatrix} \cdot \begin{pmatrix} v & 0 & 0 \\ 0 & w & 0 \\ x & y & z \end{pmatrix} \right] = \lambda \cdot \begin{pmatrix} av & 0 & 0 \\ 0 & bw & 0 \\ cv+ex & dw+ey & ez \end{pmatrix}$$

$$(\lambda A) \cdot B = \begin{pmatrix} \lambda a & 0 & 0 \\ 0 & \lambda b & 0 \\ \lambda c & \lambda d & \lambda e \end{pmatrix} \cdot \begin{pmatrix} v & 0 & 0 \\ 0 & w & 0 \\ x & y & z \end{pmatrix} = \begin{pmatrix} \lambda av & 0 & 0 \\ 0 & \lambda bw & 0 \\ \lambda cv+\lambda ex & \lambda dw+\lambda ey & \lambda ez \end{pmatrix}''$$

$$A(\lambda B) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & e \end{pmatrix} \cdot \begin{pmatrix} \lambda v & 0 & 0 \\ 0 & \lambda w & 0 \\ \lambda x & \lambda y & \lambda z \end{pmatrix} = \overset{''}{\begin{pmatrix} \lambda av & 0 & 0 \\ 0 & \lambda bw & 0 \\ \lambda cv+\lambda ex & \lambda dw+\lambda ey & \lambda ez \end{pmatrix}}$$

$$(A \cdot B)\lambda = \begin{pmatrix} av & 0 & 0 \\ 0 & bw & 0 \\ cv+ex & dw+ey & ez \end{pmatrix}'' \cdot \lambda$$

(5) what about the set of all 3×3 matrices of the form: $\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & * & * \end{pmatrix}$

well $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are both elements of this set...

but $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is not.

therefore this set does not fulfill the properties of a k-algebra.

(6) if A is an algebra, then the opposite algebra A^{op} is defined on the same underlying vector space i.e. as a set of elements $A = A^{op}$.
But the multiplication in A^{op} is defined as:

$$\forall \; a, b \in A^{op} \qquad \underset{\text{multiplication in } A^{op}}{a \cdot b} := \underset{\text{multiplication in } A}{b \cdot a}$$

then A^{op} is also an algebra.

Remark: Take $\mathcal{B} = \{b_1, b_2 \dots b_n\}$ a basis of the underlying vectorspace of the k-algebra A, then every $a \in A$ is a linear combination of the b_i $i=1 \dots n$.
So if we take $a, a' \in A$ two arbitrary elements:

$$a = \sum_{i=1}^{n} \lambda_i b_i \qquad a' = \sum_{i=1}^{n} \lambda_i' b_i \qquad \text{for } \lambda_i, \lambda_i' \in k \quad i=1 \dots n$$

then their product must satisfy:

$$a \cdot a' = \left(\sum_{i=1}^{n} \lambda_i b_i\right) \cdot \left(\sum_{i=1}^{n} \lambda_i' b_i\right) = \sum_{i,j=1}^{n} \lambda_i \lambda_j' b_i b_j$$

this means, that if we specify how to multiply any two basis elements then multiplication in the k-algebra is completely determined.

Recall: A quiver Q is a quadruple (Q_0, Q_1, s, t) consisting of the following data:

- Q_0 is a set of vertices
- Q_1 is a set of arrows
- $s: Q_1 \to Q_0$ is a map, which sends an arrow to its starting point
- $t: Q_1 \to Q_0$ is a map, which sends an arrow to its end point

we represent an arrow $\alpha \in Q_1$ by drawing it from its start to its end point, like so:

$s(\alpha) \xrightarrow{\alpha} t(\alpha)$

Def let Q be a quiver. Given two paths $c = (i \mid \alpha_1, \alpha_2, \dots \alpha_r \mid j)$, $c' = (j \mid \alpha_1', \alpha_2', \dots \alpha_{r'}' \mid k)$ with $j = t(c) = s(c')$ we denote by $c \cdot c'$ the <u>concatenation of two paths</u> given by:

$$c \cdot c' = (i \mid \alpha_1, \alpha_2, \dots \alpha_r, \alpha_1', \dots \alpha_{r'}' \mid k)$$

With this new definition we can start defining multiplication for paths. Together with our remark before we can use this to construct a useful k-algebra, namely the path algebra.

Def (4.5) let Q be a quiver. We define the <u>path algebra kQ</u> of Q as the algebra with basis comprised of all paths in the quiver Q and multiplication of two basis elements c, c' given by:

$$c \cdot c' = \begin{cases} c \cdot c' & , \text{ if } s(c') = t(c) \\ 0 & \text{otherwise} \end{cases}$$

therefore the product of any two elements in the path algebra is determined by:

$$\left(\sum_c \lambda_c \cdot c\right) \cdot \left(\sum_{c'} \lambda_{c'} \cdot c'\right) = \sum_{c, c'} \lambda_c \cdot \lambda_{c'} \cdot c \cdot c'$$

Lemma (4.3) the unity element of a path algebra kQ is given by the sum of all constant paths:

$$1 = \sum_{i\in Q_0} e_i$$

proof let $a \in kQ$ then $a = \sum_c \lambda_c \cdot c$ for $\lambda_c \in k$ then

$$a\cdot\Big(\sum_{i\in Q_0} e_i\Big) = \sum_{i\in Q_0} \underbrace{\Big(\sum_c \lambda_c\cdot c\Big)e_i}_{}$$

since $c\cdot e_i = c$ if $t(c)=i$ then only path where $t(c)=i$ remain

$$\sum_{i\in Q_0}\Big(\sum_c \lambda_c\cdot c\Big)e_i = \sum_{i\in Q_0}\sum_{\substack{c\\ t(c)=i}} \lambda_c\cdot c$$

since every vertex in Q_0 appears in the sum, then every path c will appear once

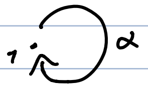
$$= \sum_c \lambda_c\cdot c = a$$

similarly: $\sum_{i\in Q_0} e_i\cdot a = \Big(\sum_{i\in Q_0} e_i\Big)\Big(\sum_c \lambda_c\cdot c\Big) = \sum_{i\in Q_0}\sum_c \lambda_c e_i\cdot c$

$$= \sum_{i\in Q_0}\sum_{\substack{c\\ s(c)=i}} \lambda_c\cdot c = \sum_c \lambda_c\cdot c = a$$

∎

examples (4.3)

(1) let Q be the quiver  $Q_0=\{1\}, Q_1=\{\alpha\}$ then the paths of Q are $e_1, \alpha, \alpha^2, \alpha^3 \ldots\ldots$ thus the path algebra kQ has basis $\{e_1, \alpha, \alpha^2, \ldots\}$ multiplication is simply: $\alpha^s\cdot\alpha^t = \alpha^{s+t}$ $s,t\in\mathbb{N}$

Then kQ is isomorphic to the algebra of polynomials over k. This can be shown through the basis elements fairly simply:

$$\varphi : kQ \longrightarrow k[x]$$
$$\sum_{n=0}^{m} \lambda_n\alpha^n \longmapsto \sum_{n=0}^{m} \lambda_n\cdot x^n$$

$$\varphi(\alpha^s\cdot\alpha^t) = \varphi(\alpha^{s+t}) = x^{s+t} = x^s\cdot x^t = \varphi(\alpha^s)\varphi(\alpha^t)$$

(2) let Q be the quiver $\overset{1}{\cdot}\xrightarrow{\alpha_1}\overset{2}{\cdot}\xrightarrow{\alpha_2}\overset{3}{\cdot}\xrightarrow{\alpha_3}\cdots\xrightarrow{\alpha_{n-1}}\overset{n}{\cdot}$
then kQ is isomorphic to the set of all upper triangular matrices $n\times n$.

Since each path in kQ is a straight path from i to j with $i\le j$ $1\le i,j\le n$ then we can uniquely denote each basis element of kQ by $c_{i,j}$ the unique path from i to j s.t.:

$$c_{ij} = \begin{cases} e_i & , i=j \\ \alpha_i\cdots\alpha_{j-1} & , i\neq j \end{cases} \qquad \text{and} \quad s(c_{ij})=i \, , \; t(c_{ij})=j$$

then

$$\varphi : kQ \longrightarrow Mat_{n\times n}\triangledown(k)$$

$$\sum_{\substack{i,j \\ 1\le i\le j\le n}} \lambda_{i,j}\cdot c_{i,j} \longmapsto \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,n} \\ & \ddots & \vdots \\ 0 & & \lambda_{n,n} \end{pmatrix}$$

note that $\varphi(\sum_{i=1}^{n} c_{i,i}) = \varphi(\sum_{i=1}^{n} e_i) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I_n$
i.e. φ sends 1_{kQ} to $1_{mat_\triangledown}$

we'll demonstrate $\varphi(a\cdot b) = \varphi(a)\cdot\varphi(b)$ by an example:

let $n=3$ $a,b\in kQ$ where $a = \sum_{1\le i\le j\le 3} \lambda_{ij}c_{ij}$ $b = \rho_{23}c_{23}$

$$a\cdot b = (\lambda_{11}c_{11} + \lambda_{12}c_{12} + \lambda_{13}c_{13} + \lambda_{22}c_{22} + \lambda_{23}c_{23} + \lambda_{33}c_{33})\cdot(\rho_{23}\cdot c_{23})$$

since $c_{ij}\cdot c_{23} \neq 0 \iff j=2$ we have:

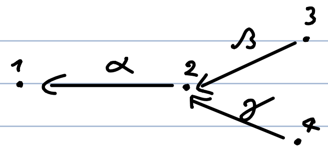
$$a\cdot b = \lambda_{12}\rho_{23}\cdot c_{12}\cdot c_{23} + \lambda_{22}\cdot\rho_{23}\cdot c_{22}\cdot c_{23} = \lambda_{12}\cdot\rho_{23}\cdot c_{13} + \lambda_{22}\cdot\rho_{23}\cdot c_{23}$$

on the other hand:

$$\varphi(a)\cdot\varphi(b) = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho_{23} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \lambda_{12}\rho_{23} \\ 0 & 0 & \lambda_{22}\rho_{23} \\ 0 & 0 & 0 \end{pmatrix} = \varphi(a\cdot b)$$

Also, we'll demonstrate an example that when 2 points can't be concatenated, we get the 0 matrix.
For example, take $a = \alpha_1 = c_{12}$, and $b = e_3 = c_{33}$
then: $a\cdot b = c_{12}\cdot c_{33} = 0$ and $\varphi(a) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\varphi(b) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
therefore $\varphi(a)\cdot\varphi(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \varphi(0) = \varphi(a\cdot b)$

(3) let Q be the quiver



then kQ is isomorphic to the set of all matrices of the form:
where $\lambda_c \in K$ are the coefficients of the path c.

$$\begin{pmatrix} \lambda_{e_1} & 0 & 0 & 0 \\ \lambda_\alpha & \lambda_{e_2} & 0 & 0 \\ \lambda_{\beta\cdot\alpha} & \lambda_\beta & \lambda_{e_3} & 0 \\ \lambda_{\alpha\gamma} & \lambda_\gamma & 0 & \lambda_{e_4} \end{pmatrix}$$

kQ has the basis:

$$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \beta\alpha, \gamma\alpha\}$$

the structure and map for this isomorphism is very similar to (2), so we won't go into detail here.

Def (4.6) let A, B be two k-algebras, then a k-linear map $f: A \to B$ is a homomorphism of algebras if:

- $f(1) = 1$
- $\forall a, a' \in A \quad f(a \cdot a') = f(a) \cdot f(a')$

We already used such homomorphisms in examples 4.3. Now we'll give you a problem where you can try to work with these maps as well:

Problem 4.5 let G be a group and let

$$kG := \{ \sum_{g\in G} \lambda_g g \mid \lambda_g \in k, \text{ finitely many } \lambda_g \text{ are non zero} \}$$

be the k-algebra with basis G and multiplication given by the group operation.
kG is called the group algebra of G.
Show that:

(1) $k\mathbb{Z}$ is isomorphic to the algebra of Laurent polynomials in one variable over k.

(2) $k(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to $k[x]/(x^n - 1)$

Solutions

(1) define $\Phi : k\mathbb{Z} \longrightarrow k[x, x^{-1}]$ ← Laurent polynomials in X

$$\sum_{n\in\mathbb{Z}} \lambda_n \cdot n \longmapsto \sum_{n\in\mathbb{Z}} \lambda_n x^n$$

we need to show that Φ is a bijective k-algebra homomorphism: let $a, b \in k\mathbb{Z}$ $\rho \in k$

$$a = \sum_{n\in\mathbb{Z}} \lambda_n \cdot n \qquad b = \sum_{n\in\mathbb{Z}} \lambda_n' \cdot n$$

- $\Phi(\rho \cdot a + b) = \Phi(\rho \cdot \sum_{n\in\mathbb{Z}} \lambda_n \cdot n + \sum_{n\in\mathbb{Z}} \lambda_n' \cdot n) = \Phi(\sum_{n\in\mathbb{Z}} (\rho\lambda_n + \lambda_n')n)$
  $= \sum_{n\in\mathbb{Z}} (\rho\lambda_n + \lambda_n') x^n = \rho \sum_{n\in\mathbb{Z}} \lambda_n x^n + \sum_{n\in\mathbb{Z}} \lambda_n' x^n = \rho \cdot \Phi(a) + \Phi(b)$
  $\Rightarrow \Phi$ is k linear
- in $k\mathbb{Z}$ the unity is 0 since in this k algebra $n \cdot m := n + m$
  $\Phi(0) = \Phi(1 \cdot 0) = 1 \cdot x^0 = 1$ ← unity in $k[x, x^{-1}]$
- $\Phi(a \cdot b) = \Phi((\sum_{n\in\mathbb{Z}} \lambda_n \cdot n) \cdot (\sum_{n\in\mathbb{Z}} \lambda_n' \cdot n)) = \Phi(\sum_{n\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} \lambda_k \cdot \lambda_{n-k}') \cdot n)$
  $= \sum_{n\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} \lambda_k \cdot \lambda_{n-k}') x^n = (\sum_{n\in\mathbb{Z}} \lambda_n x^n) \cdot (\sum_{n\in\mathbb{Z}} \lambda_n' x^n) = \Phi(a) \cdot \Phi(b)$
  $\Rightarrow \Phi$ is a k-algebra homomorphism.
- $\Phi(a) = \Phi(b) \Leftrightarrow \sum_{n\in\mathbb{Z}} \lambda_n x^n = \sum_{n\in\mathbb{Z}} \lambda_n' x^n$
  $\Leftrightarrow \lambda_n = \lambda_n' \ \forall n \in \mathbb{Z} \Leftrightarrow \sum_{n\in\mathbb{Z}} \lambda_n \cdot n = \sum_{n\in\mathbb{Z}} \lambda_n' \cdot n \Leftrightarrow a = b$
  $\Rightarrow \Phi$ is injective
- let $\sum_{n\in\mathbb{Z}} a_n x^n \in k[x, x^{-1}]$ then $a_n \in k \ \forall n \in \mathbb{Z}$
  so then $\sum_{n\in\mathbb{Z}} a_n \cdot n \in k\mathbb{Z}$ and $\Phi(\sum_{n\in\mathbb{Z}} a_n \cdot n) = \sum_{n\in\mathbb{Z}} a_n x^n$
  $\Rightarrow \Phi$ is surjective

■

(2) First off, let us recall that for any $p(x) \in k[x]$ using long devision, one can find $q(x), r(x) \in k[x]$ s.t.

$$p(x) = (x^n - 1) q(x) + r(x) \quad \text{where:}$$

$r(x) = \sum_{i=0}^{n-1} \lambda_i x^i$, hence $k[x]/(x^n - 1) \cong \{ \sum_{i=0}^{n-1} \lambda_i x^i \mid \lambda_i \in k \}$

next, note that in $k[x]/(x^n-1)$, $x^m \equiv x^{m \pmod n}$ $\forall m \in \mathbb{N}$
hence, in multiplying two elements in $k[x]/(x^n-1)$
we can replace $x^i \cdot x^j$ by $x^{i+j \pmod n}$, so:

$$\left(\sum_{i=0}^{n-1} \lambda_i x^i\right)\left(\sum_{j=0}^{n-1} \lambda_j' x^j\right) = \sum_i \sum_j \lambda_i \lambda_j' x^{i+j} = \sum_{k=0}^{n-1}\left(\sum_{i+j=k \bmod n} \lambda_i \lambda_j'\right) x^k$$

$$= \sum_{k=0}^{n-1} (\lambda_0 \lambda_k' + \lambda_1 \lambda_{k-1}' + \dots + \lambda_k \lambda_0' + \lambda_{k+1} \lambda_{n-1}' + \dots + \lambda_{n-1} \cdot \lambda_{k+1}')\, x^k$$

similarly for $a, b \in k(\mathbb{Z}/n\mathbb{Z})$ where
$a = \sum_{i=0}^{n-1} \lambda_i \cdot i$ $\quad b = \sum_{j=0}^{n-1} \lambda_j' j$ then:

$$a \cdot b = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \lambda_i \lambda_j' \cdot (i+j) \bmod n = \sum_{k=0}^{n-1}\left(\sum_{i+j=k \bmod n} \lambda_i \lambda_j'\right) k$$

$$= \sum_{k=0}^{n-1} (\lambda_0 \lambda_k' + \lambda_1 \lambda_{k-1}' + \lambda_2 \lambda_{k-2}' + \dots + \lambda_k \lambda_0' + \lambda_{k+1} \lambda_{n-1}' + \dots + \lambda_{n-1} \lambda_{k+1}')\, k$$

so now let
$$\psi : k(\mathbb{Z}/n\mathbb{Z}) \longrightarrow k[x]/(x^n-1)$$
$$\sum_{i=0}^{n-1} \lambda_i \cdot i \longmapsto \sum_{i=0}^{n-1} \lambda_i x^i$$

we need to show that ψ is a bijective k-algebra homomorphism. let $a, b \in k(\mathbb{Z}/n\mathbb{Z})$ $\rho \in k$
$a = \sum_{i=0}^{n-1} \lambda_i \cdot i$ $\quad b = \sum_{i=0}^{n-1} \lambda_i' \cdot i$

- $\psi(\rho \cdot a + b) = \psi\left(\rho \cdot \sum_{i=0}^{n-1} \lambda_i i + \sum_{i=0}^{n-1} \lambda_i' i\right) = \psi\left(\sum_{i=0}^{n-1} (\rho \lambda_i + \lambda_i') i\right)$
$= \sum_{i=0}^{n-1} (\rho \lambda_i + \lambda_i') x^i = \rho \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=0}^{n-1} \lambda_i' x^i = \rho \cdot \psi(a) + \psi(b)$
$\Rightarrow \psi$ is k linear

- Again 0 is the unity of $k(\mathbb{Z}/n\mathbb{Z})$
$\psi(0) = x^0 = 1 \leftarrow$ the unity in $k[x]/(x^n-1)$

- $\psi(a \cdot b) = \psi\left(\left(\sum_{i=0}^{n-1} \lambda_i \cdot i\right) \cdot \left(\sum_{i=0}^{n-1} \lambda_i' \cdot i\right)\right)$
$= \psi\left(\sum_{i=0}^{n-1} (\lambda_0 \lambda_i' + \lambda_1 \lambda_{i-1}' + \lambda_2 \lambda_{i-2}' + \dots + \lambda_i \lambda_0 + \lambda_{i+1} \lambda_{n-1}' + \dots + \lambda_{n-1} \lambda_{i+1}') i\right)$
$= \sum_{i=0}^{n-1} (\lambda_0 \lambda_i' + \lambda_1 \lambda_{i-1}' + \dots + \lambda_i \lambda_0 + \lambda_{i+1} \cdot \lambda_{n-1}' + \dots + \lambda_{n-1} \lambda_{i+1}')\, x^i$
$= \left(\sum_{i=0}^{n-1} \lambda_i x^i\right) \cdot \left(\sum_{i=0}^{n-1} \lambda_i' x^i\right) = \psi(a) \cdot \psi(b)$
$\Rightarrow \psi$ is a k-algebra homomorphism

- $\psi(a) = \psi(b) \iff \sum_{i=0}^{n-1} \lambda_i \cdot x^i = \sum_{i=0}^{n-1} \lambda_i' x^i \iff \lambda_i = \lambda_i' \; \forall i=0..n-1$
  $\iff \sum_{i=0}^{n-1} \lambda_i i = \sum_{i=0}^{n-1} \lambda_i' i \iff a = b$

- let $\sum_{i=0}^{n-1} a_i x^i \in k[x]/_{(x^n-1)}$ then $a_i \in k \; \forall i=0..n-1$
  so $\sum_{i=0}^{n-1} a_i \cdot i \in k(\mathbb{Z}/n\mathbb{Z})$ and $\psi(\sum_{i=0}^{n-1} a_i \cdot i) = \sum_{i=0}^{n-1} a_i x^i$
  $\Rightarrow \psi$ is surjective

■

Def (4.7) let B be a k-vector subspace of A then B is a subalgebra if B contains 1 and $\forall b, b' \in B \quad b \cdot b' \in B$

Prop (4.4) If $I \subseteq A$ is a nilpotent ideal of A s.t. the Algebra $A/I \cong k \times k \times \dots \times k$ then $I = \operatorname{rad} A$

proof we already know that $I \subseteq \operatorname{rad} A$ from Corollary 4.2
since k is a field, we know $0, k$ are its sole ideals, hence maximal ideals of $k \times \dots \times k$ are:
$0 \times k \times k \dots \times k \;,\; k \times 0 \times k \times \dots \times k \;\dots\; k \times \dots \times k\, 0 \times k \;,\; k \times \dots \times k \times 0$
$\Rightarrow \operatorname{rad}(A/I) = 0$ since the radical ideal is the intersection of all the maximal ideals by definition.

consider
$$\begin{aligned} \pi : A &\to A/I \\ a &\mapsto a+I \end{aligned}$$

let $a \in \operatorname{rad}(A)$ then from lemma 4.1 we know $\forall b \in A$ $1-ba$ has an iverse $c \in A$, then:

$1+I = \pi(1) = \pi(c \cdot (1-ba)) = \pi(c)\pi(1-ba) = \pi(c) \cdot (1 - \pi(b) \cdot \pi(a))$
which means $(1 - \pi(b) \cdot \pi(a))$ has an inverse in A/I
then again by lemma 4.1 $\pi(a) \in \operatorname{rad}(A/I)$
but we've show $\operatorname{rad}(A/I) = 0 \Rightarrow \pi(a) = 0 \Rightarrow a \in I$
therefore $\operatorname{rad} A \subseteq I$

■

Corollary (4.5) If Q is a quiver without oriented cycles, then rad kQ is the ideal generated by all arrows in Q.

A path of the form
$$\begin{tikzcd} i \arrow[r, "\alpha_1"] & \cdot \arrow[r, "\alpha_2"] & \cdot \cdots \arrow[r, "\alpha_{l-1}"] & \cdot \arrow[lll, bend left, "\alpha_l"] \end{tikzcd}$$
given by $(i \,|\, \alpha_1, \alpha_2, \ldots \alpha_{l-1}, \alpha_l \,|\, i)$, is an oriented cycle.

proof We denote by R_Q the ideal generated by all arrows in Q. We let l be the largest integer s.t. Q will contain a path of length l, i.e. any product of $l+1$ arrows will be 0.
This means $R_Q^{l+1} = 0$ hence R_Q is a nilpotent ideal.
Also $\{e_i + R_Q \mid i \in Q_0\}$ is a basis for kQ/R_Q
so $kQ/R_Q \cong k \times \ldots \times k$ ← number of copies of k is $|Q_0|$
then by Prop 4.4 $R_Q = \text{rad}(kQ)$ ∎

Remark (4.6) It's important in this corollary that Q fulfills the condition of having no oriented cycles. For this quiver Q : $\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$ for example

the path algebra kQ is isomorphic to the polynomials $k[x]$. And since every linear polynomial $x-a$, for $a \in k$, generates a maximal ideal, we see that rad $k[x] = 0$.

4.3 Modules

In this section we'll define modules over a Ring R with 1. We'll also present some examples and finish with an example of a morphism between two modules over a path algebra.

Def (4.8) let R be a ring with $1 \neq 0$ a <u>right R-module M</u> is an abelian group together with a binary operation, called the right R-action:

$$\begin{aligned} M \times R &\longrightarrow M \\ (m, r) &\longmapsto m \cdot r \end{aligned}$$

s.t. $\forall m_1, m_2 \in M$ and $\forall r_1, r_2 \in R$ we have that:

(1) $(m_1 + m_2) r_1 = m_1 r_1 + m_2 r_2$

(2) $m_1 (r_1 + r_2) = m_1 r_1 + m_1 \cdot r_2$

(3) $m_1 \cdot (r_1 \cdot r_2) = (m_1 \cdot r_1) r_2$

(4) $m_1 \cdot 1 = m_1$

a <u>left R-module</u> is defined by multiplying the elements of M from the left and following the axioms (1) ... (4) accordingly.

examples (4.5)

(1) If $I \subseteq R$ is a right ideal, then I is a right R-module, where the right R-action is given by multiplication in R. In particular, the ideal generated by $a \in R$ namely $a \cdot R = \{ a \cdot r / r \in R \}$ is a right R-module

(2) If Q is a quiver and $A = kQ$ is its path algebra then for any vertex $i \in Q_0$ we can define an A-module $S(i)$ whose abelian group is the one dimensional k-vector space generated by $\{e_i\}$ and whose A action is given by:

$$\forall c \in A \qquad m \cdot e_i \cdot c = \begin{cases} m \cdot e_i & , \text{ if } c = e_i \\ 0 & , \text{ otherwise} \end{cases}$$

let's check that $S(i)$ fullfils the module axioms. In this case it's enough to only use two paths $c, c' \in A$ since other cases follow by k-Linearity of A and $S(i)$. let $m_1, m_2 \in k$ $c, c' \in A$ then:

- $(m_1 e_i + m_2 e_i) \cdot c = (m_1 + m_2) e_i \cdot c = \begin{cases} (m_1 + m_2) e_i & , \text{ if } c = e_i \\ 0 & , \text{ otherwise} \end{cases}$

$m_1 e_i c + m_2 e_i c = \begin{cases} m_1 e_i + m_2 e_i = (m_1 + m_2) e_i & \text{if } c = e_i \\ 0 & , \text{ otherwise} \end{cases}$

$\Rightarrow (m_1 e_i + m_2 e_i) c = m_1 e_i c + m_2 e_i c$

- $m_1 e_i (c+c') = \begin{cases} m_1 e_i, & c+c'=e_i \Leftrightarrow c=e_i, c'=0 \text{ or } c=0, c'=e_i \\ 2m_1 e_i, & c=c'=e_i \\ 0, & \text{otherwise} \end{cases}$

  $m_1 e_i c + m e_i c' = \begin{cases} m_1 e_i, & c=0, c'=e_i \text{ or } c=e_i, c'=0 \\ 2m_1 e_i, & c=c'=e_i \\ 0, & \text{otherwise} \end{cases}$

  $\Rightarrow m_1 e_i (c+c') = m_1 e_i c + m_1 e_i c'$

- $m_1 e_i (c\cdot c') = \begin{cases} m_1 e_i, & \text{if } c\cdot c'=e_i \Leftrightarrow c=c'=e_i \\ 0, & \text{otherwise} \end{cases}$

  $(m_1 e_i c)c' = \begin{cases} (m_1 e_i)c', & \text{if } c=e_i \\ 0 & \text{otherwise} \end{cases} = \begin{cases} m_1 e_i, & \text{if } c'=e_i \\ 0, & \text{otherwise} \end{cases}$

  $\Rightarrow m_1 e_i (c\cdot c') = (m_1 e_i c)c'$

- $m_1 e_i \cdot 1 = m_1 e_i \sum_{j\in Q_0} e_j = m_1 (\sum_{j\in Q_0} e_i e_j) = m_1 e_i$

(3) If Q is a quiver and $A = kQ$ is its path algebra then for any arrow $i \xrightarrow{\alpha} j$ in Q_1, we can define an A-module $M(\alpha)$ whose abelian group is equal to the two dimensional k-vector space generated by $\{e_i, \alpha\}$ and whose right A-action is given by:

$\forall\ \lambda_i, \lambda_\alpha \in k \quad \forall\ c$ a path in A

$$(\lambda_i e_i + \lambda_\alpha \alpha)c = \lambda_i e_i c + \lambda_\alpha \alpha c = \begin{cases} \lambda_i e_i, & \text{if } c=e_i \\ \lambda_\alpha \alpha, & \text{if } c=e_j \\ \lambda_i \alpha, & \text{if } c=\alpha \\ 0, & \text{otherwise} \end{cases}$$

we see that this right action coincides with the multiplication we defined on the path algebra. For example: $\lambda_i e_i \cdot \alpha := \lambda_i \alpha$ makes sense since e_i is the constant path at i and α is an arrow from i to j, which means their concatenation $i \xrightarrow{\alpha} j$ is a ain α.
Showing that $M(\alpha)$ fulfills the module axioms is very similar to how we proceeded in (2) so we won't go into further detail here.

$S(i)$ and $M(\alpha)$ might seem familiar to you. You may have seen them as $S(i)$, the simple quiver rep. and as j. This is no coincidence. They'll likely appear later on again.

Def (4.9) A module M is said to be generated by the elements $m_1, m_2, \dots, m_s$ if for every $m \in M$ there exist $a_i \in R$ s.t. $m = a_1 m_1 + a_2 m_2 + \dots + a_s m_s$
M is called finitely generated if it is generated by a finite set of elements.

Remark If M is generated by $m_1, m_2, \dots m_s$ then
$M = m_1 \cdot R + m_2 \cdot R + \dots + m_s R$
for example: the ideal $a \cdot R$ is finitely generated by only one element $a \in R$.

Def (4.10) let M, N be two R-modules.
A map $h : M \longrightarrow N$ is called a morphism of R-modules if $\forall m, m' \in M$, $\forall a \in R$ we have:

- $h(m + m') = h(m) + h(m')$
- $h(ma) = h(m) \cdot a$

The kernel of h is the set $\operatorname{Ker}(h) = \{ m \in M \mid h(m) = 0 \}$
the image of h is the set $\operatorname{Im}(h) = \{ h(m) \mid m \in M \}$
and the cokernel of h is $\operatorname{coker}(h) = N / \operatorname{Im}(h)$

Remark If A is a k-algebra then a morphism of two A-modules is also a homomorphism of the underlying k-vector spaces and thus a linear map.

Prop (4.8) If $h : M \to N$ is a morphism of A-modules, then $\operatorname{Ker}(h)$, $\operatorname{Im}(h)$, $\operatorname{coker}(h)$ are A-modules.

proof let $m_1, m_2 \in \operatorname{Ker}(h)$ $\lambda_1, \lambda_2 \in A$ then:

- $h(m_1 + m_2) = h(m_1) + h(m_2) = 0 + 0 = 0 \quad \Rightarrow m_1 + m_2 \in \operatorname{Ker}(h)$
- $h(\lambda_1 m_1) = \lambda_1 h(m_1) = \lambda_1 \cdot 0 = 0 \quad \Rightarrow \lambda_1 \cdot m_1 \in \operatorname{Ker}(h)$

- $(m_1 + m_2)\lambda_1 = m_1 \lambda_1 + m_2 \lambda_2$
- $m_1(\lambda_1 + \lambda_2) = m_1 \lambda_1 + m_1 \lambda_2$
- $m_1(\lambda_1 \cdot \lambda_2) = (m_1 \cdot \lambda_1)\lambda_2$
- $m_1 \cdot 1 = m$

} follows directly from the fact that $Ker(h) \subseteq M$ an A-module

$\Rightarrow$ $Ker(h)$ is an A-module

■

the proofs for $Im(h)$ and $Coker(h)$ are very similar and easy. So we won't bore you with this here.

example (4.6) let $A = kQ$ be a path algebra. And let $S(j)$ and $M(\alpha)$ be A-modules as defined in examples (4.5)(2) and (3), where $j \in Q_0$ and $\alpha \in Q_1$ with $t(\alpha) = j$. Then there is a morphism:

$$\begin{aligned} h : S(j) &\longrightarrow M(\alpha) \\ m e_j &\longmapsto m\alpha \end{aligned}$$

let's check that h is indeed a morphism of modules:
let $m_1, m_2 \in k$ $\lambda \in k$ $c \in A$

- $h(\lambda m_1 e_j + m_2 e_j) = h((\lambda m_1 + m_2)e_j) = (\lambda m_1 + m_2)\alpha = \lambda m_1 \alpha + m_2 \alpha$
  $= \lambda \cdot h(m_1 e_j) + h(m_2 e_j)$ $\Rightarrow$ h is k linear

- $h(m_1 e_j c) = \begin{cases} h(m_1 e_j) = m_1 \alpha & , \text{if } c = e_j \\ h(0) = 0 & , \text{otherwise} \end{cases}$

  $h(m_1 e_j) \cdot c = m_1 \alpha \cdot c \begin{cases} m_1 \alpha , & \text{if } c = e_j \\ 0 , & \text{otherwise} \end{cases}$

  $\Rightarrow h(m_1 e_j c) = h(m_1 e_j)c$

therefore h is a morphism of modules