The Diagrammatic Theory part V

Soergel's Categorification Theorem

Plan

PART 1

Last time, we constructed a functor $\mathscr{F}$ from the diagrammatic category $\mathscr{H}_{BS}$ to $\mathbb{BS}$Bim. By constructing an explicit basis for the R – linear category $\mathscr{H}_{BS}$, we can study the subcategory $\mathscr{F}(\mathscr{H}_{BS})$ of $\mathbb{BS}$Bim, which will turn out to be full and faithful.

PART 2

We have seen that Kar($\mathbb{BS}$Bim) = $\mathbb{S}$Bim.

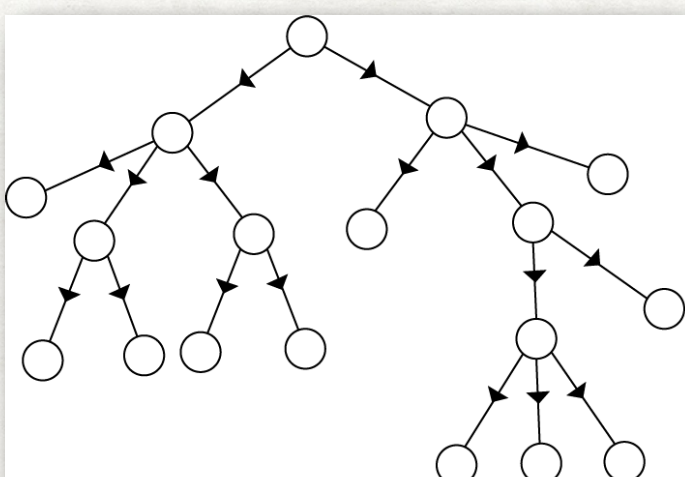
In the section, we will look at something like Kar($\mathscr{H}_{BS}$) and hope for the best !

Recall: **The Bruhat Order**

Given a Coxeter system (W, S), and two elements x and y such that $l(x) < l(y)$ and $\exists t \in W$ such that $xt = y$

We denote this partial relation between x and y by $x \to y$.

This partial relation gives W the structure of a directed tree, and we can extended transitively by following $\to$



Also recall U_0, U_1, D_0, D_1 and the defect of a subexpression:

Given a subexpression $\underline{e} = (e_1, \ldots, e_n)$ of $\underline{w} = (s_1, \ldots, s_n)$, where each $e_i = 0$ or 1

we can decorate $\underline{e}$ with a sequence of length n of elements from $\{U_0, U_1, D_0, D_1\}$ as follows:

if $s_1 \ldots s_i > s_1 \ldots s_{i-1}$, we put U_{e_i} in the i^{th} position, otherwise we put D_{e_i}.

The defect is defined to be $\#U_0 - \#D_0$

Example:

Consider the sequence (s, s, s) and the subexpressions $(1,0,0)$ and $(1,0,1)$.

then the corresponding decorations are (U_1, D_0, D_0) and (U_1, D_0, D_1) repectively.

Overview of Rex Moves:

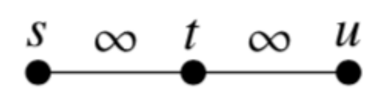
Definition: The rex graph corresponding to $w \in W$ is the graph whose vertices are the distinct reduced expressions of w, such that two vertices are connected by an edge if they are related by braid relations.

Remark: By Matsumoto's Theorem, all rex graphs are connected.

Definition: A rex move corresponding to a path in the rex graph of $w \in W$ is defined to be the composition of the corresponding morphisms in the diagrammatic category $\mathscr{H}_{BS}$.

In other words, it is the sequence of $2m_{st}$-valent morphisms in $\mathscr{H}_{BS}$ corresponding to the braid relations appearing in the given path.

Example 10.23. Suppose $W = \langle s, u, t \rangle$ with Coxeter graph



Here we give the rex graph for the element $w = sutsu$:

$$\begin{tikzcd}[column sep=small]
& (s,u,t,s,u) \arrow[dl, dash] \arrow[dr, dash] & \\
(u,s,t,s,u) \arrow[dr, dash] & & (s,u,t,u,s) \arrow[dl, dash] \\
& (u,s,t,u,s) &
\end{tikzcd}$$

This is an example of a disjoint cycle, emerging from the application of two disjoint braid relations. Note that $(s,u,t,u,s) \to (s,u,t,s,u) \to (u,s,t,s,u)$ and $(s,u,t,u,s) \to (u,s,t,u,s) \to (u,s,t,s,u)$ are two distinct paths in the rex graph from (s,u,t,u,s) to (u,s,t,s,u).

A quick detour:

Given $w \in W$, where W is a rank 3 finite Coxeter group, then there is a cycle in the rex graph of the longest element.

We call such a cycle a Zamolodchikov cycle

Slogan: No need for 4-color relations

Libedinsky's light leaves

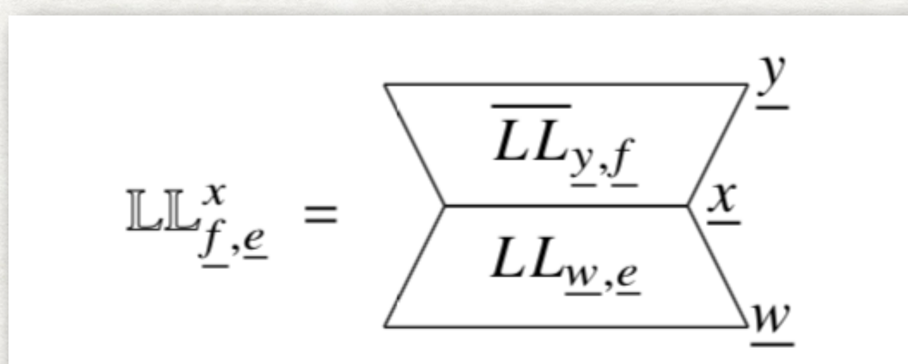
Given an expression $\underline{w}$ and a subexpression $\underline{e}$ with target $\underline{w}^{\underline{e}} = x$, we will construct a morphism $LL_{\underline{w},\underline{e}} \in \mathscr{H}_{BS}$ from $\underline{w}$ to $\underline{x}$, where $\underline{x}$ is a reduced expression for x. Moreover, $\deg(LL_{\underline{w},\underline{e}})$ will be the defect of $\underline{e}$.

Given such a morphism $LL_{\underline{w},\underline{e}} \in \mathscr{H}_{BS}$, one can flip the diagram to get a morphism

$$\overline{LL}_{\underline{w},\underline{e}} \in \mathscr{H}_{BS}$$

Now if $\underline{e} \subseteq \underline{w}$ and $\underline{f} \subseteq \underline{y}$ such that $\underline{w}^{\underline{e}} = \underline{y}^{\underline{f}} = x$,

$$\text{we define } \mathbb{LL}^{x}_{\underline{f},\underline{e}} := \overline{LL}_{\underline{y},\underline{f}} \circ LL_{\underline{w},\underline{e}} \in \mathscr{H}_{BS}, \text{ taking } \underline{w} \to \underline{y}.$$



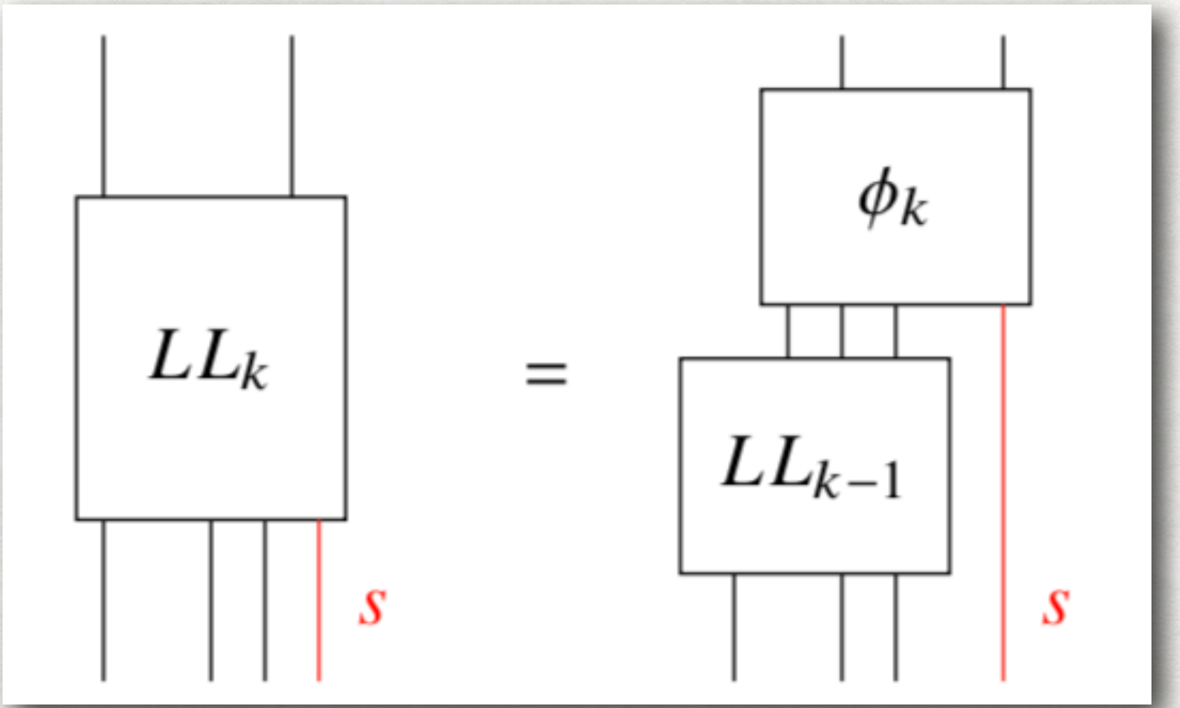
Actual construction of the leaves

Step 0: We start by defining $LL_{\emptyset,\emptyset}$ to be the identity in R.

Step k: assuming we know how LL_{k-1} maps $\underline{w}_{\leq k-1} \to \underline{x}_{k-1}$,

we will define LL_k as a morphism $\underline{w}_k \to \underline{x}_k$ by:

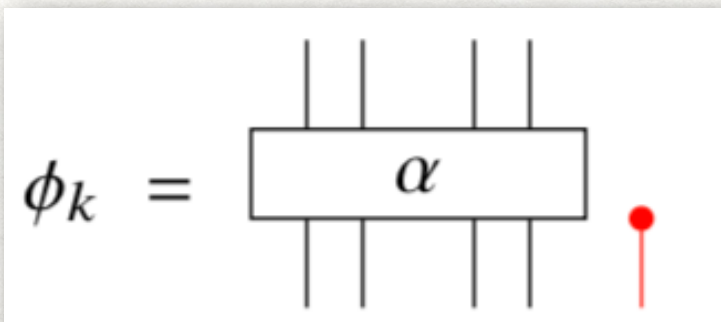
$LL_k := \phi_k \circ (LL_{k-1} \otimes \mathrm{id}_s)$, where ϕ_k will be defined in the next slides.



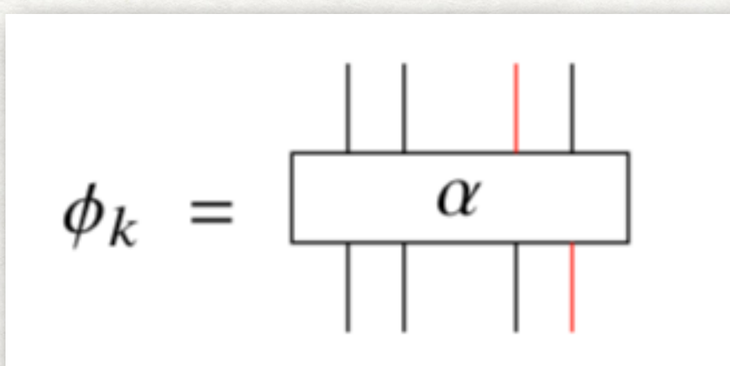
Describing ϕ_k

ϕ_k depends on the decoration given to e_k . it is given diagrammatically as follows:

Case 1: U_0



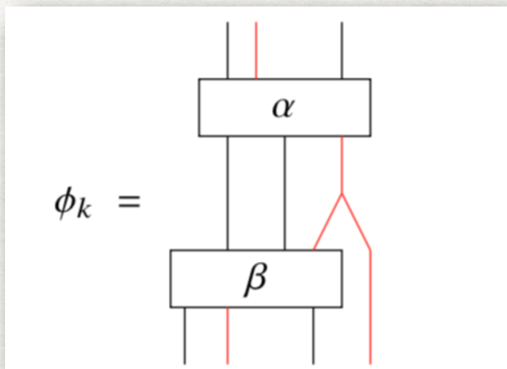
Case 2: U_1



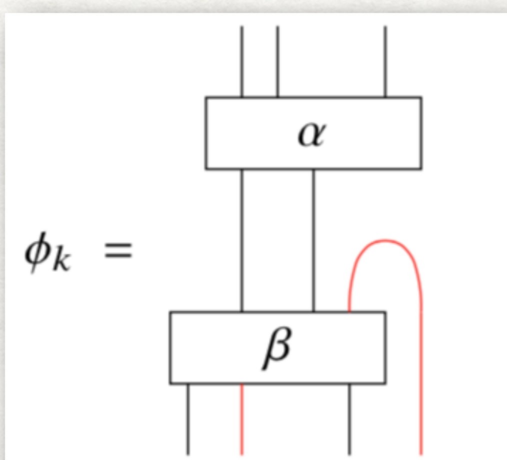
-Both $\alpha's$ appearing in the diagrams are rex moves

-They come from the fact that although $x_{k-1} = x_k$, we don't necessarily have $\underline{x}_{k-1} = \underline{x}_k$.

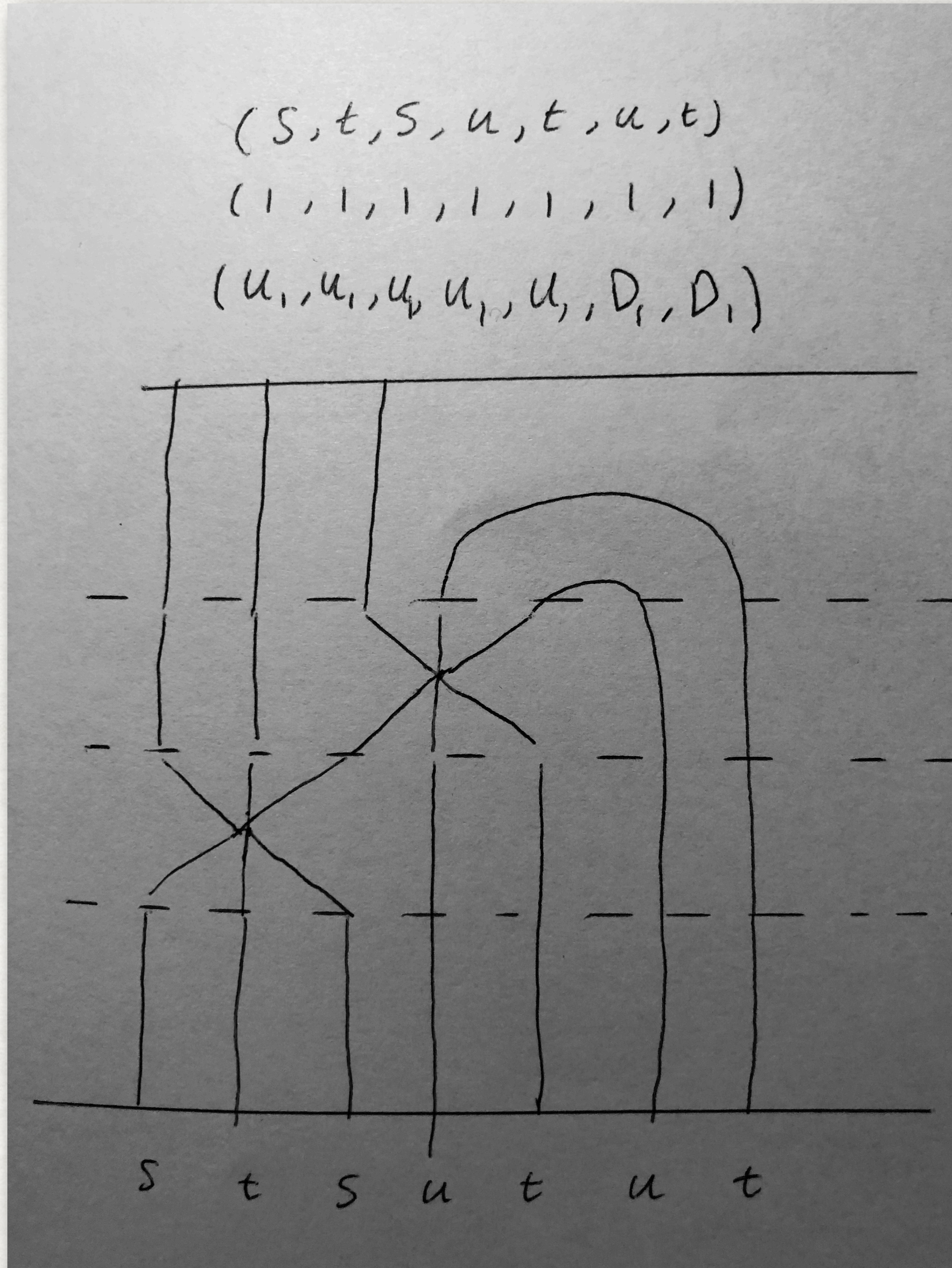
Case 3:D_0



Case 4:D_1



Examples

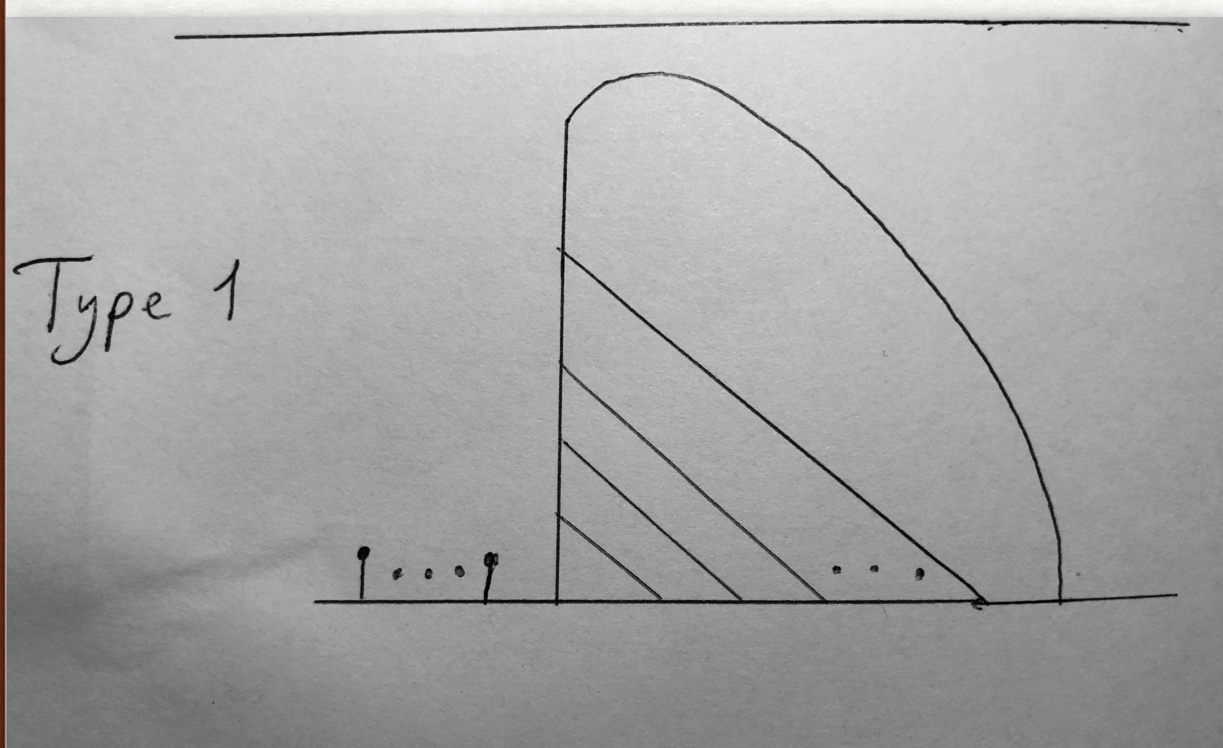


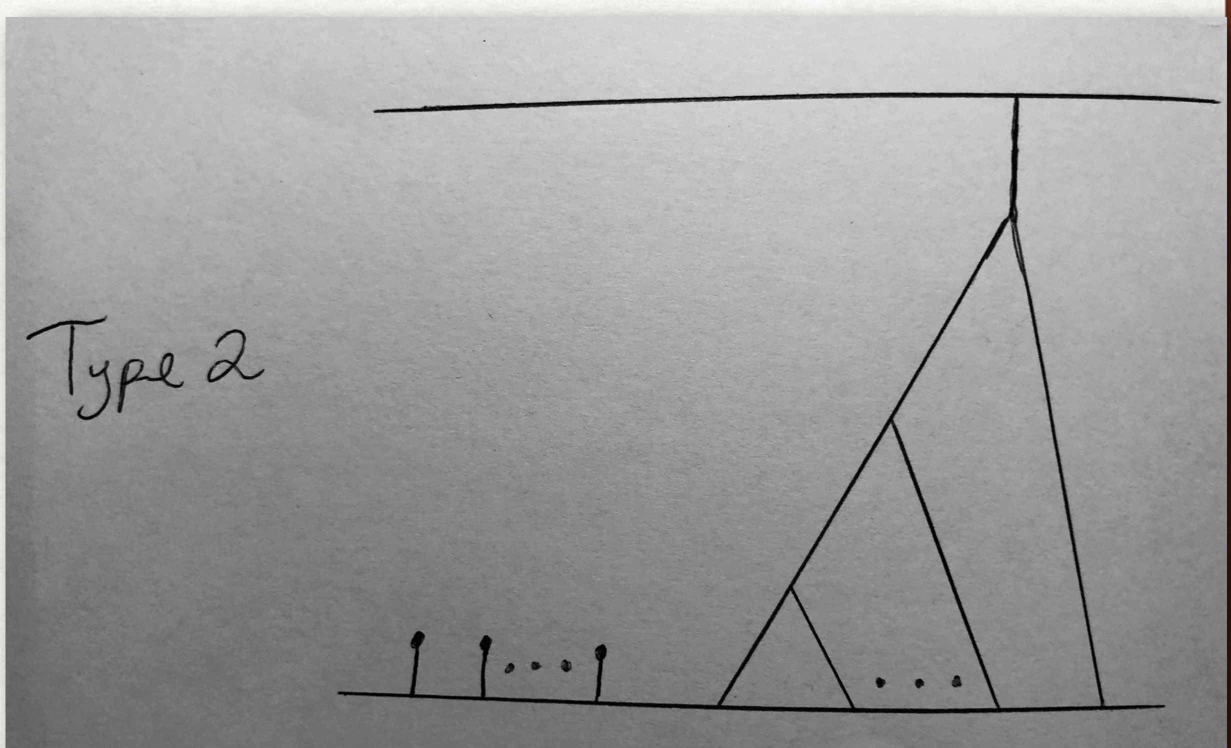
Examples Continued

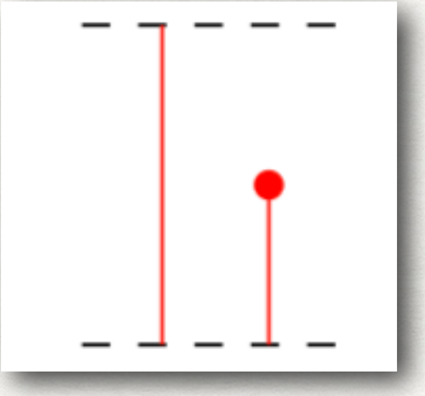
Let $(s, s, \ldots, s)$ be of length m, then any light leaf is of the form:

$A_1 \otimes \ldots \otimes A_k \otimes B$, where each A_i is of type 1 and B is of type 1 or 2 .

Moreover, we need to have $m = \#\{\text{sources in } B\} + \sum_i \#\{\text{sources in } A_i\}$.







Is not a light leaf!

Theorem: (Double leaves are a basis)

Fix expressions $\underline{w}, \underline{y}$ in (W, S). Let $\mathbb{LL}_{\underline{w},\underline{y}}$ denote the collection $\{\mathbb{LL}^{x}_{\underline{f},\underline{e}}\}_{((\underline{w},\underline{e}),(\underline{y},\underline{f}),x)}$ such that the triples satisfy $\underline{w}^{\underline{e}} = \underline{y}^{\underline{f}} = x$, then the family $\mathbb{LL}_{\underline{w},\underline{y}}$ is a basis for $\mathrm{Hom}_{\mathscr{H}_{BS}}(\underline{w}, \underline{y})$ as a right (or left) module.

An Exercise:

The Soergel Hom formula together with results form the earlier sections show that

$$\underline{\mathrm{rk}}\mathrm{Hom}^{\bullet}(\mathrm{BS}(\underline{w}), \mathrm{BS}(\underline{y})) = |\mathbb{LL}_{\underline{w},\underline{y}}|$$

Theorem 5.27 (Soergel Hom formula [167]). *For any two Soergel bimodules B, B', the graded Hom space $\mathrm{Hom}^{\bullet}_{\mathbb{S}\mathrm{Bim}}(B, B')$ is free as a left graded R-module with graded rank $(\mathrm{ch}(B), \mathrm{ch}(B'))$:*

$$\underline{\mathrm{rk}}\,\mathrm{Hom}^{\bullet}_{\mathbb{S}\mathrm{Bim}}(B, B') = (\mathrm{ch}(B), \mathrm{ch}(B')). \qquad (5.30)$$

Here, $(-,-)$ denotes the standard form on H *(see Definition 3.13), and* $\underline{\mathrm{rk}}$ *denotes the graded rank (see the end of Sect. 4.1). It is also free as a right graded R-module with the same graded rank.*

Corollary:

The previously defined functor $\mathscr{F}$ from the diagrammatic category $\mathscr{H}_{BS}$ to $\mathbb{BS}$Bim is an equivalence of categories!!

Missing ingredient for a proof:

$-\mathscr{F}$ is faithful

A proof of this is implicit in the proof of the previous theorem!

Constructing $\mathscr{H}$

A graded category, is on in which all Hom$'s$ are $\mathbb{Z}$-graded and such that:

$$\text{Hom}^i(Y,Z) \circ \text{Hom}^j(X,Y) \subseteq \text{Hom}^{i+j}(X,Z)$$

To a pre-additive category $\mathscr{C}$ with a shift functor (1), one can constuct a graded category $\mathscr{C}^{gr}$ by defining the new Hom's to be:

$$\text{Hom}^\bullet(X,Y) := \bigoplus_{k\in\mathbb{Z}} \text{Hom}^k(X,Y), \text{where} \text{Hom}^k(X,Y) := \text{Hom}_{\mathscr{C}}(X, Y(k))$$

The functor $(-)^{gr}$ has a left adjoint! It can be defined as follows:

Objects in $\mathscr{C}^{sh}$ are formal objects $X(n)$ and morphisms are

$$\text{Hom}_{\mathscr{C}^{sh}}(X(n), Y(m)) := \text{Hom}^{m-n}_{\mathscr{C}}(X,Y)$$

Now we can construct $\mathscr{H}$ from $\mathscr{H}_{BS}$ in three steps:

1) Apply $(-)^{sh}$ to $\mathscr{H}_{BS}$

2) Take the additive closure

3) Take the Karoubian envelope

Definition :(The Diagrammatic Hecke Category)

$\mathscr{H}$ is taken to be the resulting category .

Theorem 11.1 (Soergel categorification theorem).

1. *For each reduced expression $\underline{w}$ the object $\underline{w} \in \mathcal{H}$ has a unique indecomposable direct summand $B_{\underline{w}}$ which does not occur as a direct summand in any shorter expression.*
2. *Let $w \in W$. If $\underline{w}$ and $\underline{w}'$ are reduced expressions for w, then $B_{\underline{w}}$ and $B_{\underline{w}'}$ are isomorphic.*[2] *We denote the isomorphism class of $B_{\underline{w}}$ by B_w.*
3. *Up to shift, any indecomposable object of $\mathcal{H}$ is isomorphic to some B_w.*
4. *The map $b_s \mapsto [s]$ for $s \in S$ induces a $\mathbb{Z}[v^{\pm 1}]$-algebra isomorphism*

$$c \colon \mathrm{H} \to [\mathcal{H}]_{\oplus} \ . \tag{11.1}$$

5. *(Soergel Hom formula) For any two objects X, Y of $\mathcal{H}$, let $x, y \in \mathrm{H}$ be the elements for which $c(x) = [X]$ and $c(y) = [Y]$. Then the graded Hom space $\mathrm{Hom}^{\bullet}_{\mathcal{H}}(X, Y)$ is free as a left graded R-module with graded rank (x, y).*

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Theorem 5.24 (Soergel's Categorification Theorem [167]). *Under the technical assumptions to be discussed in Sect. 5.6, we have the following results.*

1. *There is a $\mathbb{Z}[v^{\pm 1}]$-algebra homomorphism*

$$c : \mathrm{H} \to [\mathbb{S}\mathrm{Bim}]_{\oplus} \tag{5.27}$$

sending $b_s \mapsto [B_s]$ for all $s \in S$.

2. *There is a bijection between W and the set of indecomposable objects of $\mathbb{S}\mathrm{Bim}$ up to shift and isomorphism:*

$$\begin{aligned} W &\xleftrightarrow{1:1} \{\text{ indec. objects in } \mathbb{S}\mathrm{Bim} \}/\simeq, (1) \\ w &\longleftrightarrow B_w . \end{aligned} \tag{5.28}$$

The indecomposable object B_w appears as a direct summand of the Bott–Samelson bimodule $\mathrm{BS}(\underline{w})$ for a reduced expression of w. Moreover, all other summands of $\mathrm{BS}(\underline{w})$ are shifts of B_x for $x < w$ in the Bruhat order.

3. *The character function $\mathrm{ch} = \mathrm{ch}_{\Delta}$ defined above descends to a $\mathbb{Z}[v^{\pm 1}]$-module homomorphism*

$$\mathrm{ch}\colon [\mathbb{S}\mathrm{Bim}]_{\oplus} \to \mathrm{H} \tag{5.29}$$

which is the inverse to c. Thus, both are isomorphisms.

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