Categorification: An introduction to $\mathfrak{sl}_n$-link homologies

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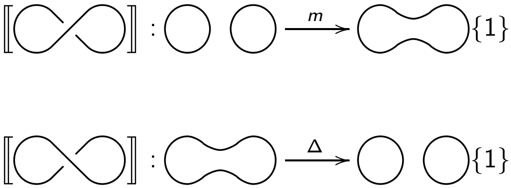


What is categorification?

Forced to reduce this presentation to one sentence, the author would choose:

Interesting integers are shadows of richer structures in categories.

The basic idea can be seen as follows. Take a "set-based" structure S and try to find a "category-based" structure $\mathcal{C}$ such that S is just a shadow of $\mathcal{C}$.

Categorification, which can be seen as "remembering" or "inventing" information, comes with an "inverse" process called decategorification, which is more like "forgetting" or "identifying".

Note that decategorification should be easy.

The underlying basic example

Take $\mathcal{C} = K$-**FinVec** for a fixed field K, i.e. objects are finite dimensional K-vector spaces $V, V', \dots$ and morphisms are K-linear maps $f\colon V \to V'$ between them. $\mathcal{C}$ categorifies $\mathbb{N}$: We can go back by taking the dimension $\dim V \in \mathbb{N}$.

What is the upshot? Note the following:

- Much information is lost if we only consider $\mathbb{N}$, i.e.

$$n = n' \Leftrightarrow V \cong V'.$$

- We have the power of linear algebra between V and V', i.e. $\hom_K(V, V')$.
- A vector space can carry additional structure.

Never forget the original structure

The structure of $\mathbb{N}$ is reflected on a "higher" level!

- The direct sum $\oplus$ and the tensor product $\otimes_K$ categorify $+$ and $\cdot$, i.e.

$$\dim(V \oplus V') = \dim V + \dim V' \text{ and } \dim(V \otimes_K V') = \dim V \cdot \dim V'.$$

- The zero vector space 0 and the field K categorify the identities, i.e.

$$V \oplus 0 \cong V \cong 0 \oplus V \text{ and } V \otimes_K K \cong V \cong K \otimes_K V.$$

- The injections and surjections categorify the order relation, i.e.

$$\exists f\colon V \hookrightarrow V' \Leftrightarrow \dim V \leq \dim V' \text{ and } \exists f\colon V \twoheadrightarrow V' \Leftrightarrow \dim V \geq \dim V'.$$

One can write down the categorified statements of other properties as "Addition and multiplication are associative and commutative" etc.

What about quantum numbers?

Enhance $\mathcal{C} = K$-**FinVec** to $\mathcal{D} = K$-**FinVec**$_{\mathrm{gr}}$, i.e. objects are $\mathbb{Z}$-graded, finite dimensional K-vector spaces $V, V', \dots$ and morphisms are degree preserving K-linear maps $f\colon V \to V'$ between them.

Define the graded dimension by

$$\mathrm{qdim}\,(V = \bigoplus_{j\in\mathbb{Z}} V^j) = \sum_{j\in\mathbb{Z}} q^j \dim V^j.$$

$\mathcal{D}$ categorifies $\mathbb{N}[q, q^{-1}]$: We can go back by taking $\mathrm{qdim}\, V \in \mathbb{N}[q, q^{-1}]$.

Example(The dual numbers)

$$A = \mathbb{Q}[X]/X^2 = \langle 1\rangle^{+1} \oplus \langle X\rangle^{-1} \text{ and } \mathrm{qdim}\, A = q + q^{-1} = [2].$$

Integer based invariants

A topological flavoured example goes back to Riemann (1857), Betti (1871) and Poincaré (1895): The Euler characteristic $\chi(X)$ of a reasonable topological space.

Noether, Hopf and Alexandroff (1925) "categorified" this invariant as follows.

If we lift $n, n' \in \mathbb{N}$ to the two K-vector spaces V, V' with dimensions $\dim V = n, \dim V' = n'$, then the difference $n - n'$ lifts to the complex

$$0 \longrightarrow V' \xrightarrow{d} V \longrightarrow 0,$$

for any linear map d and V in even homology degree.
Iterate: If we have already lifted n to the complex C and n' to C', then we can lift $n - n'$ to the cone

$$\Gamma(\phi\colon C \to C')\colon \cdots \to C_i \oplus C'_{i-1} \xrightarrow{\tilde{d}} C_{i+1} \oplus C'_i \to \dots .$$

The Euler characteristic is a shadow

For each (singular) chain complex

$$(C(X), d_*) = \cdots \xrightarrow{d_{-2}} C_{-1}(X,\mathbb{Q}) \xrightarrow{d_{-1}} C_0(X,\mathbb{Q}) \xrightarrow{d_0} C_1(X,\mathbb{Q}) \xrightarrow{d_1} \cdots$$

define

$$\mathrm{tdim}\,(C(X)) = \sum_{i\in\mathbb{Z}} t^i \dim(\mathrm{ker}\ d_i/\mathrm{im}\ d_{i-1}) = \sum_{i\in\mathbb{Z}} t^i \underbrace{\dim H_i(X,\mathbb{Q})}_{b_i(X)}.$$

For example

$$\mathrm{tdim}\,(C(S^1)) = 1 + t.$$

Conclusion (Noether): The $(C(X), d_*)$ categorifies $\chi(X)$ by taking $t = -1$.

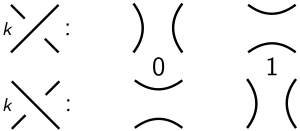
Well-known upshots

We note the following observations.

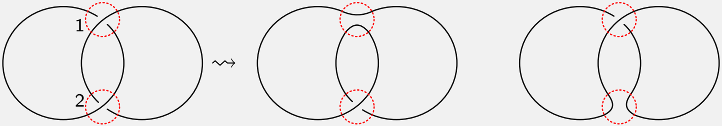
- The homology extends to a functor and provides information about continuous maps as well.
- Again, chain maps tell how some complexes are related.
- The space $H_i(X, \mathbb{Q})$ is a $\mathbb{Q}$-vector space and $b_i(X)$ is just a number: More information of X is encoded.
- Singular homology works for all topological spaces and the homological Euler characteristic can be defined for a huge class of spaces.
- More sophisticated constructions like multiplication in cohomology provide even more information.
- Not the main point, but: The $H_i(X, \mathbb{Q})$ are better invariants than the $b_i(X)$.

Crossing resolutions

Given a diagram of a link L_D with ordered crossings $\mathrm{cr}_1, \ldots, \mathrm{cr}_n$.
The 0-resolution and the 1-resolution of the k-th crossing are locally defined by



Example(Hopf link resolutions part 1)

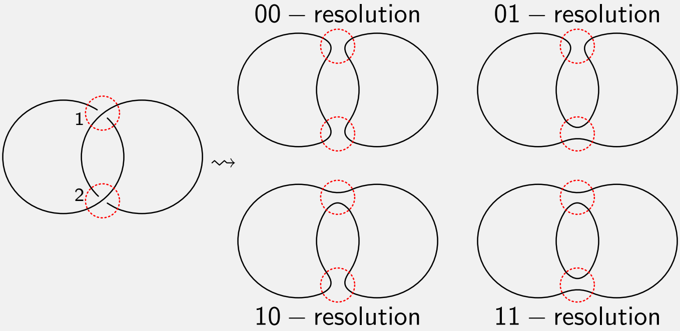


1 – resolution of cr_1 $\quad$ 0 – resolution of cr_2

Link resolutions

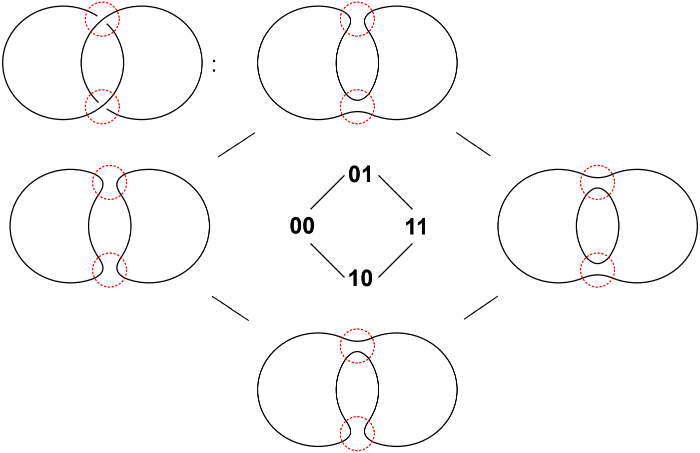
Given a diagram of a link L_D with ordered crossings $\mathrm{cr}_1, \dots, \mathrm{cr}_n$.
An element $\vec{r} = r_1 \dots r_n \in \{0,1\}^n$, i.e. ordered strings of 0's and 1's of length n, is called an (abstract) $\vec{r}$-resolution of L_D. A $\vec{r}$-resolution of L_D is inductively defined by applying the r_k-resolution to the crossing cr_k.

Example(Hopf link resolutions part 2)



The Khovanov cube

Given a diagram of a link L_D with ordered crossings $\mathrm{cr}_1, \ldots, \mathrm{cr}_n$. Define the Khovanov cube of L_D to be the 1-skeleton of the n-dimensional hypercube $[0,1]^n$ whose vertices are replaced by the corresponding $\vec{r}$-resolution of L_D.



The famous Jones polynomial

Let L_D be a diagram of an oriented link. Set $[2] = q + q^{-1}$ and

n_+ = number of crossings  n_- = number of crossings 

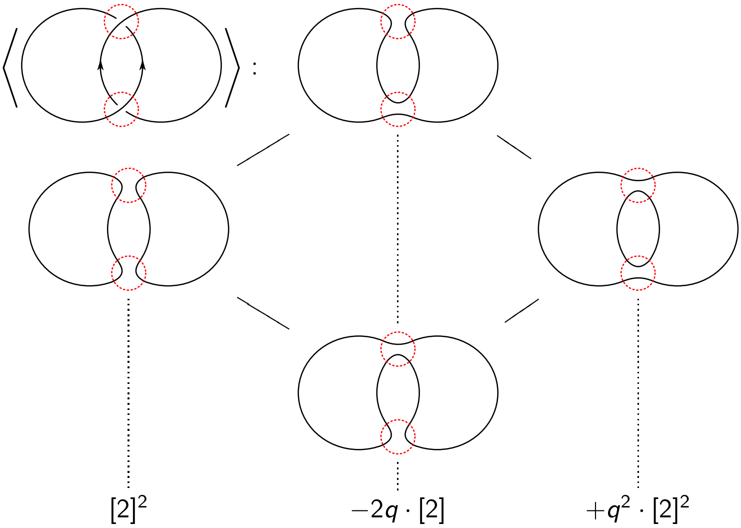
Definition/Theorem(Jones 1984, Kauffman 1987)

The bracket polynomial of the diagram L_D (without orientations) is a polynomial $\langle L_D \rangle \in \mathbb{Z}[q, q^{-1}]$ given by the following recursive local rules.

- $\langle \emptyset \rangle = 1$ (normalization).
- $\langle$$\rangle = \langle)\ (\rangle - q\langle$$\rangle$ (recursion step 1a).
- $\langle$$\rangle = \langle$$\rangle - q\langle)\ (\rangle$ (recursion step 1b).
- $\langle \bigcirc \amalg L_D \rangle = [2] \cdot \langle L_D \rangle$ (recursion step 2).
- $[2]J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$ (Re-normalization).

The polynomial $J(\cdot) \in \mathbb{Z}[q, q^{-1}]$ is an invariant of oriented links.

Exempli gratia



Thus, $J(\mathbf{Hopf}) = q + q^5 \neq [2] = J(\bigcirc\bigcirc)$, i.e. the Hopf link is not trivial!

Please, fasten your seat belts!

Let's categorify everything!

The way-leading idea

Idea(Khovanov 1999)

Recall: A quantum number $Q \in \mathbb{N}[q, q^{-1}]$ can be categorified using graded vector spaces and an integer $z \in \mathbb{Z}$ can be categorified using chain complexes.
Khovanov: There should be a chain complex of graded vector spaces for each L_D, denoted by $\mathbf{Kh}(L_D)$, with graded i-th homology group $\bigoplus_{j\in\mathbb{Z}} H_i^j(L_D)$, such that

$$\mathrm{tqdim}\left(\mathbf{Kh}(L_D)\right) = \sum_{i,j\in\mathbb{Z}} t^i q^j \dim H_i^j(L_D)$$

$$\chi_{\mathrm{q}}\left(\mathbf{Kh}(L_D)\right) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim H_i^j(L_D) = [2]J(L_D),$$

i.e. taking the graded Euler characteristic $\chi_{\mathrm{q}}(\cdot)$ gives the Jones polynomial.

Khovanov: Categorified rules

Set $\mathcal{C} = \mathbf{Ch}_b(\mathbb{Q} - \mathbf{FinVec}_{\mathrm{gr}})$, i.e. the category of bounded chain complexes of finite dimensional, graded, $\mathbb{Q}$-vector spaces. Define the bracket $[\![L_D]\!]$ and the Khovanov complex $\mathbf{Kh}(L_D)$ as objects of $\mathcal{C}$ by categorifying the rules for the Jones polynomial. That is we discuss now how to fill the following table (ordered from "easy" to "hard").

Uncategorified world	Categorified world
$\langle \emptyset \rangle = 1$	?
$[2]J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$	?
$\langle \bigcirc \amalg L_D \rangle = [2] \cdot \langle L_D \rangle$	?
$\langle \diagup\!\!\!\!\diagdown \rangle = \langle)\,(\rangle - q \langle \asymp \rangle$	?
$\langle \diagdown\!\!\!\!\diagup \rangle = \langle \asymp \rangle - q \langle)\,(\rangle$	?

Normalization is simple

The normalization and re-normalization are simple. We only need to set a value for the empty diagram and shift in homology degree

$$\Big(\bigoplus_{i,j\in\mathbb{Z}} H_i^j(L_D)\Big)\langle k\rangle = \bigoplus_{i,j\in\mathbb{Z}} H_{i-k}^j(L_D)$$

and quantum degree

$$\Big(\bigoplus_{i,j\in\mathbb{Z}} H_i^j(L_D)\Big)\{k\} = \bigoplus_{i,j\in\mathbb{Z}} H_i^{j-k}(L_D).$$

Uncategorified world	Categorified world
$\langle\emptyset\rangle = 1$	$[\![\emptyset]\!] = 0 \longrightarrow \mathbb{Q} \longrightarrow 0$
$[2]J(L_D) = (-1)^{n_-}q^{n_+-2n_-}\langle L_D\rangle$	$\mathbf{Kh}(L_D) = [\![L_D]\!]\langle n_-\rangle\{n_+ - 2n_-\}$
$\langle \bigcirc \amalg L_D\rangle = [2]\cdot\langle L_D\rangle$	?
$\langle \diagup\!\!\!\!\diagdown \rangle = \langle)\,(\rangle - q\langle \asymp \rangle$	?
$\langle \diagdown\!\!\!\!\diagup \rangle = \langle \asymp \rangle - q\langle)\,(\rangle$	?

The dual numbers as basic piece

Recall the $A = \mathbb{Q}[X]/X^2 = \langle 1 \rangle^{+1} \oplus \langle X \rangle^{-1}$. It categorifies quantum [2], i.e $\mathrm{qdim}\, A = q + q^{-1} = [2]$. Moreover, recall that the second recursion rule gives

$$\langle \bigcirc \rangle = [2] \cdot \langle \emptyset \rangle = [2] \Rightarrow \langle \bigcirc\bigcirc \rangle = [2] \cdot \langle \bigcirc \rangle = [2]^2 \text{ etc.}$$

Recall that $\cdot$ is categorified by $\otimes$. Thus, Khovanov proposed:

Uncategorified world	Categorified world
$\langle \emptyset \rangle = 1$	$[\![\emptyset]\!] = \; 0 \longrightarrow \mathbb{Q} \longrightarrow 0$
$[2]J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$	$\mathbf{Kh}(L_D) = [\![L_D]\!]\langle n_- \rangle \{n_+ - 2n_-\}$
$\langle \bigcirc \amalg L_D \rangle = [2] \cdot \langle L_D \rangle$	$[\![\bigcirc \amalg L_D]\!] = A \otimes [\![L_D]\!]$
$\langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle$	?
$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$	?

The cone complex

Given two chain complexes $A = (A_i, a_i)$ and $B = (B_i, b_i)$ and a chain morphisms $d\colon A \to B$ between them, define the cone

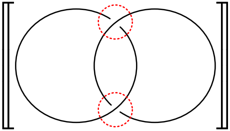
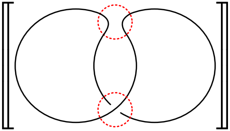
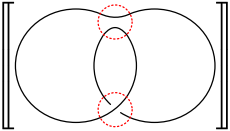
$$\Gamma(d\colon A \to B) = \ \ldots \longrightarrow A_i \oplus B_{i-1} \xrightarrow{\begin{pmatrix} a_i & 0 \\ d_i & -b_i \end{pmatrix}} A_{i+1} \oplus B_i \longrightarrow \ldots$$

Assume that we know what d_* is. Then Khovanov proposed:

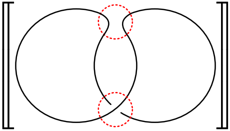
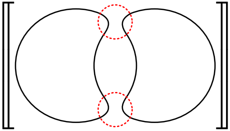
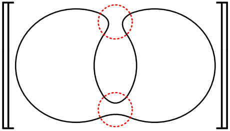
Uncategorified world	Categorified world
$\langle \emptyset \rangle = 1$	$[\![\emptyset]\!] = \ 0 \longrightarrow \mathbb{Q} \longrightarrow 0$
$[2]J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$	$\mathbf{Kh}(L_D) = [\![L_D]\!] \langle n_- \rangle \{n_+ - 2n_-\}$
$\langle \bigcirc \amalg L_D \rangle = [2] \cdot \langle L_D \rangle$	$[\![\bigcirc \amalg L_D]\!] = A \otimes [\![L_D]\!]$
$\langle \slashoverback \rangle = \langle)(\rangle - q \langle \asymp \rangle$	$[\![\slashoverback]\!] = \Gamma(d_*\colon [\![)(]\!] \to [\![\asymp]\!]\{1\})$
$\langle \backoverslash \rangle = \langle \asymp \rangle - q \langle)(\rangle$	$[\![\backoverslash]\!] = \Gamma(d_*\colon [\![\asymp]\!] \to [\![)(]\!]\{1\})$

A recursive procedure

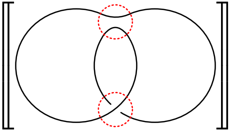
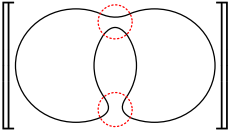
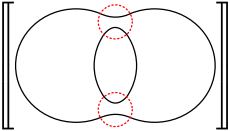
The calculation of the Khovanov complex is recursive, i.e. replacing the first crossing with the cone rule gives (I skip to denote Γ):

 $= d_{*-}:$  $\rightarrow$  $\{1\}$

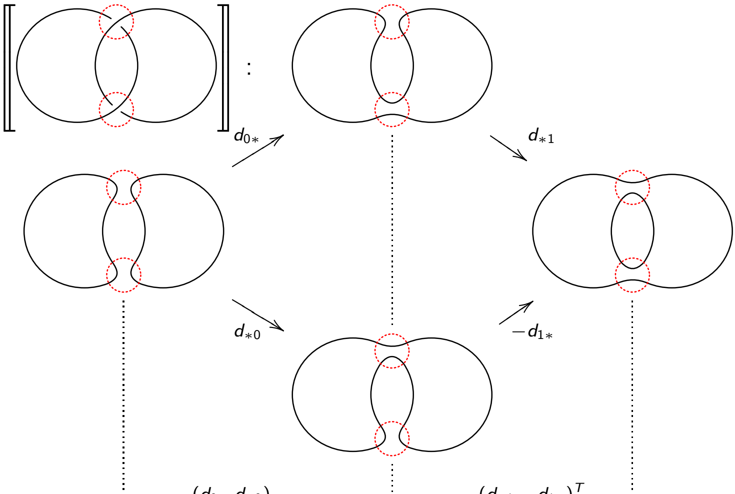
Replacing the second crossing gives

 $= d_{0*}:$  $\rightarrow$  $\{1\}$

and

 $= d_{1*}:$  $\rightarrow$  $\{1\}$

Only the differentials are missing



$$A^{\otimes 2} \xrightarrow{(d_{0*}, d_{*0})} A\{1\} \oplus A\{1\} \xrightarrow{(d_{*1}, -d_{1*})^T} A^{\otimes 2}\{2\}$$

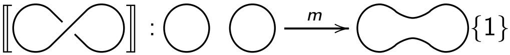
Only the differentials are missing!

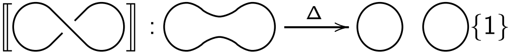
Fixing two cases suffice

Observation

For the Jones polynomial $J(\cdot)$, due to the recursive procedure, it was enough to fix the value on the empty diagram. The same is true for $[\![\cdot]\!]$ on the level of chain groups. Thus, only the differentials are missing.
But again, due to the recursive procedure, it suffices to fix the differentials in all cases where L_D has exactly one crossing.

There are (up to isotopies) exactly two diagrams L_D with one crossing, namely





It is a Frobenius algebra

Definition

The algebra $A = \mathbb{Q}[X]/X^2$ is a Frobenius algebra with multiplication m_A given by

$m_A \colon A \otimes A \to A, m_A(1 \otimes 1) = 1, m_A(1 \otimes X) = X = m_A(X \otimes 1)$ and $m_A(X \otimes X) = 0$

and comultiplication Δ_A given by

$$\Delta_A \colon A \to A \otimes A, \Delta_A(1) = 1 \otimes X + X \otimes 1 \text{ and } \Delta_A(X) = X \otimes X.$$

Both are of degree -1 (e.g. $\deg X = -1$ and $\deg \Delta_A(X) = \deg X \otimes X = -2$). Thus, define

$$m \colon A \otimes A \to A\{1\}, m = m_A \quad \text{and} \quad \Delta \colon A \to A \otimes A\{1\}, \Delta = \Delta_A.$$

These are maps of degree 0.

The Khovanov complex

Definition/Theorem(Khovanov 1999)

Let L_D be an oriented link diagram with ordered crossings $\mathrm{cr}_1, \dots, \mathrm{cr}_n$.
Define $[\![L_D]\!]$ and $\mathbf{Kh}(L_D)$ recursively using the following categorified rules.

- $[\![\emptyset]\!] = \; 0 \longrightarrow \mathbb{Q} \longrightarrow 0$ (normalization).
- $[\![\times_{+}]\!] = \Gamma(d_* \colon [\![)(]\!] \to [\![\asymp]\!]\{1\})$ (recursion step 1a).
- $[\![\times_{-}]\!] = \Gamma(d_* \colon [\![\asymp]\!] \to [\![)(]\!]\{1\})$ (recursion step 1b).
- $[\![\bigcirc \amalg L_D]\!] = A \otimes [\![L_D]\!]$ (recursion step 2).
- $\mathbf{Kh}(L_D) = [\![L_D]\!]\langle n_- \rangle\{n_+ - 2n_-\}$ (Re-normalization).

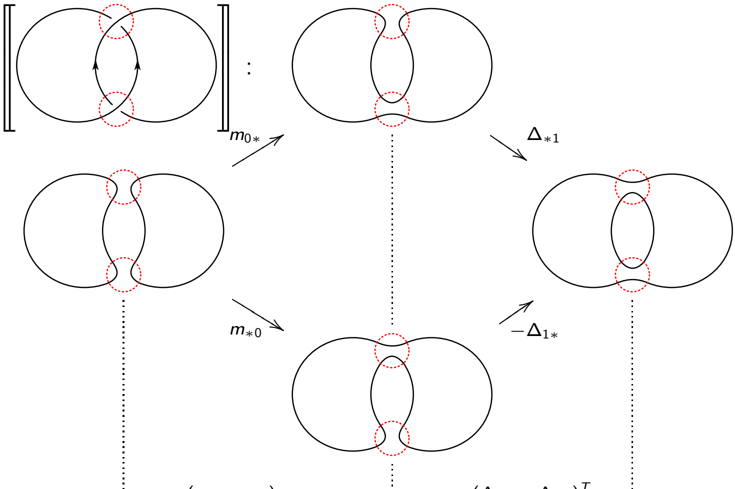
The Khovanov complex of L_D does not depend (up to chain isomorphisms) on the ordering of the crossings and is well-defined as a chain complex (aka $d^2 = 0$).
It is an invariant of oriented links, i.e.

$$L_D \sim \tilde{L}_D \Rightarrow \mathbf{Kh}(L_D) = \mathbf{Kh}(\tilde{L}_D) \text{ as objects of } \mathcal{C}_h \text{ (the homotopy category).}$$

Moreover, the Khovanov complex categorifies the Jones polynomial, i.e.

$$\chi_{\mathrm{q}}(\mathbf{Kh}(L_D)) = [2]J(L_D).$$

Exempli gratia



$$A^{\otimes 2} \xrightarrow{(m_{0*}, m_{*0})} A\{1\} \oplus A\{1\} \xrightarrow{(\Delta_{*1}, -\Delta_{1*})^T} A^{\otimes 2}\{2\}$$

$$\mathrm{tqdim}\, \mathbf{Kh}(\mathbf{Hopf}) = 1 + q^2 + t^2q^4 + t^2q^6 \overset{t=-1}{=} [2](q+q^5) = [2]J(\mathbf{Hopf})$$

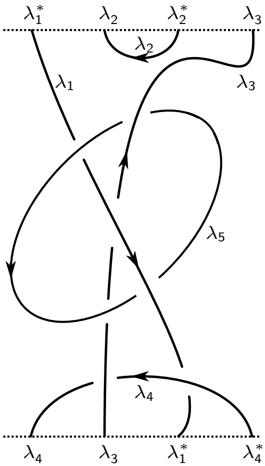
This is better than the Jones polynomial

- It is functorial (in this formulation only up to a sign).
- Khovanov's construction can be extended to a categorification of the HOMFLY-PT polynomial.
- Kronheimer and Mrowka showed that Khovanov homology detects the unknot. This is still an open question for the Jones polynomial.
- Rasmussen obtained from the homology an invariant that "knows" the slice genus and used it to give a combinatorial proof of the Milnor conjecture.
- Rasmussen also gives a way to combinatorial construct exotic $\mathbb{R}^4$.
- The categorification is not unique, e.g. the so-called "odd Khovanov homology" differs over $\mathbb{Q}$.
- Before I forget: It is a strictly stronger invariant.

After Khovanov lots of other homologies of "Khovanov-type" were discovered. So we need to understand this better.

A tangle is an intertwiner

Let $\mathfrak{g}$ be any classical Lie algebra. Denote by λ_i, μ_j the $\mathbf{U}_q(\mathfrak{g})$-representation of highest weight V_{λ_i}, V_{μ_j}. Let T_D be a diagram of a, λ_i, μ_j-colored, oriented tangle.



$\rightsquigarrow$

$$V^*_{\lambda_1} \otimes V_{\lambda_2} \otimes V^*_{\lambda_2} \otimes V_{\lambda_3}$$

$$\uparrow f(T_D)$$

$$V_{\lambda_4} \otimes V_{\lambda_3} \otimes V^*_{\lambda_1} \otimes V^*_{\lambda_4}$$

Representation theory does the trick!

Definition(Reshetikhin-Turaev 1990)

For a diagram of a colored, oriented tangle T_D with b bottom and t top points and each pair of tuples $(\lambda_1, \dots, \lambda_b), (\mu_1, \dots, \mu_t)$ define a certain $\mathbf{U}_q(\mathfrak{g})$-intertwiner

$$f(T_D)\colon V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_b} \to V_{\mu_1} \otimes \cdots \otimes V_{\mu_t}.$$

Theorem(Reshetikhin-Turaev 1990)

The $\mathbf{U}_q(\mathfrak{g})$-intertwiner $f(T_D)$ is an invariant of T_D.

Corollary(Reshetikhin-Turaev 1990)

In the case of colored, oriented links L_D we have

$$f(L_D)\colon \mathbb{Q}(q) \to \mathbb{Q}(q),\ 1 \mapsto P_{\mathbf{RT}}(L_D) \in \mathbb{Z}[q, q^{-1}],$$

that is each configuration as above gives a polynomial invariant of oriented links!

This is powerful!

Example

We have the following list of examples!

- Let $\mathfrak{g} = \mathfrak{sl}_2$. If we restrict to the $\mathbf{U}_q(\mathfrak{sl}_2)$-vector representation $\mathbb{Q}^2$, then the Reshetikhin-Turaev polynomial $P_{\mathbf{RT}}(\cdot)$ is the Jones or $\mathfrak{sl}_2$-polynomial.
- Let $\mathfrak{g} = \mathfrak{sl}_n$. If we restrict to the $\mathbf{U}_q(\mathfrak{sl}_n)$-vector representation $\mathbb{Q}^n$, then the Reshetikhin-Turaev polynomial $P_{\mathbf{RT}}(\cdot)$ is the $\mathfrak{sl}_n$-polynomial.
- But the Reshetikhin-Turaev polynomial is much more generalize than all of them and "explains" them using one concept.
- This can be also done in the root of unity case $q = \exp(2\pi i \frac{k}{n})$, i.e. it is connected to the Witten-Reshetikhin-Turaev invariants of 3-Manifolds.

Moral: A lot of link polynomials are special instances of symmetries of the quantum groups $\mathbf{U}_q(\mathfrak{g})$!

The quantum algebra $\mathbf{U}_q(\mathfrak{sl}_m)$

Definition

For $m \in \mathbb{N}_{>1}$ the quantum special linear algebra $\mathbf{U}_q(\mathfrak{sl}_m)$ is the associative, unital $\mathbb{Q}(q)$-algebra generated by $K_i^{\pm 1}$ and E_i and F_i, for $i = 1, \dots, m-1$, subject the following relations.

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q - q^{-1}},$$

$$K_iE_j = q^{(\epsilon_i,\alpha_j)}E_jK_i,$$

$$K_iF_j = q^{-(\epsilon_i,\alpha_j)}F_jK_i,$$

$$E_i^2E_j - [2]E_iE_jE_i + E_jE_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,$$

$$E_iE_j - E_jE_i = 0, \qquad \text{else},$$

$$F_i^2F_j - [2]F_iF_jF_i + F_jF_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,$$

$$F_iF_j - F_jF_i = 0, \qquad \text{else}.$$

The idempotented version

Definition(Beilinson-Lusztig-MacPherson)

For each $\vec{k} \in \mathbb{Z}^{m-1}$ adjoin an idempotent $1_{\vec{k}}$ (think: projection to the $\vec{k}$-weight space!) to $\mathbf{U}_q(\mathfrak{sl}_m)$ and add some relations, e.g.

$$1_{\vec{k}} 1_{\vec{k}'} = \delta_{\vec{k},\vec{k}'} 1_{\vec{k}} \text{ and } K_{\pm i} 1_{\vec{k}} = q^{\pm \vec{k}_i} 1_{\vec{k}} \text{ (no } K's \text{ anymore!)}.$$

The idempotented quantum special linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_m) = \bigoplus_{\vec{k},\vec{k}' \in \mathbb{Z}^{m-1}} 1_{\vec{k}} \mathbf{U}_q(\mathfrak{sl}_m) 1_{\vec{k}'}.$$

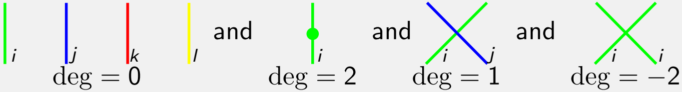
Its lower part $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ is the subalgebra of only F's.

An important fact: The $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ has the "same" representation theory as $\mathbf{U}_q(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ suffices to describe it.

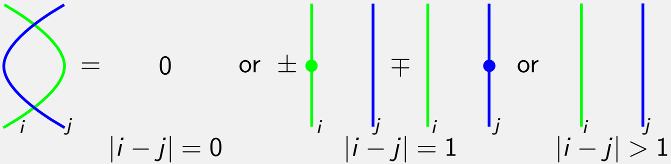
The cyclotomic KL-R algebra

Definition(Khovanov-Lauda, Rouquier 2008/2009)

The KL-R algebra R_m associated to a number $m > 1$ is defined to be the free, graded $\mathbb{Q}$-algebra generated by horizontal and vertical stacking of $k \in \{1, \ldots, m-1\}$-colored idempotents, dotted lines and crossings



where multiplication is defined by vertical stacking of diagrams if colors and number of strands match and zero otherwise modulo some relations like



There is a cyclotomic quotient R_Λ associated to a $\mathbf{U}_q(\mathfrak{sl}_m)$-highest weight Λ.

It is a categorification!

Theorem(Khovanov-Lauda, Rouquier)

We have

$$\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m) \cong K_0^{\oplus}(R_m) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q).$$

Theorem(Brundan-Kleshchev, Lauda-Vazirani, Webster,...>2008)

Let V_Λ be the $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module of highest weight Λ. We have (as $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-modules)

$$V_\Lambda \cong K_0^{\oplus}(R_\Lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q).$$

Question

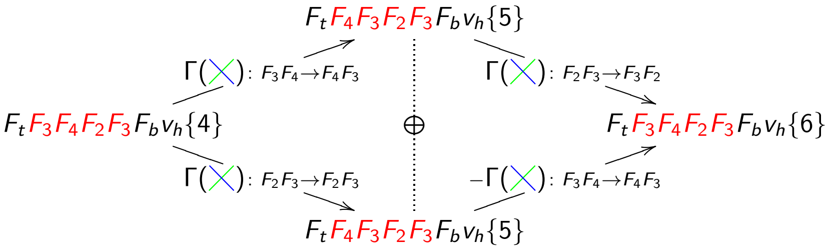
Reshetikhin-Turaev used $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory for the link-polynomials. Can we use R_m or R_Λ to obtain Khovanov homology?

That is what I am doing

Theorem(Short answer)

Yes, but for the $\mathfrak{sl}_n$-link homology I need categorified $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory. Not quite "categorified" RT-link polynomials...

Very roughly: Use so-called categorified q-skew Howe duality to express a link diagram L_D as a certain string of only $F_i^{(j)}$'s. Obtain a complex as



This, under categorified q-skew Howe duality, gives the $\mathfrak{sl}_n$-link homology.

There is still much to do...

Thanks for your attention!