Today

Soergel calculus for an arbitrary Coxeter system

Recap Geometric Realization

- (W, S) Coxeter system
- $\{\alpha_s\}$ $\mathbb{R}$-spans V. Symmetric bilinear form on V: $(\alpha_s, \alpha_t) = -\cos\frac{\pi}{m_{st}}$
- W acts on V: $s(v) = v - 2(v, \alpha_s)\alpha_s$. Faithful
- Let $R = \mathrm{Sym}\, V$. W acts on R
- $R^s = \{f \in R: s \cdot f = f\}$ is a subring
- **Chevalley–Shephard–Todd**: R^s is a polynomial ring; R is a graded finite-rank free R^s-module
- Splitting: $R = R^s \oplus R^s\alpha_s \cong R^s \oplus R^s(-2)$

Recap Bott-Samelson Bimodules

- **Demazure operators**: $\partial_s\colon R \to R^s(-2),\ f \mapsto \frac{f - s(f)}{\alpha_s}$
- Let $B_s = R \otimes_{R^s} R(1)$ and $\mathrm{BS}(\underline{w}) = B_{s_1} \cdots B_{s_n} = R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_n}} R(n)$
- $\mathrm{BS}(\underline{w})$'s are graded free left/right R-modules of finite ranks
- $\mathbb{BS}\mathrm{Bim}$: objects are $\mathrm{BS}(\underline{w})$'s; $\mathrm{Hom}(B, B') = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{R\text{-gbim}}\big(B, B'(k)\big)$

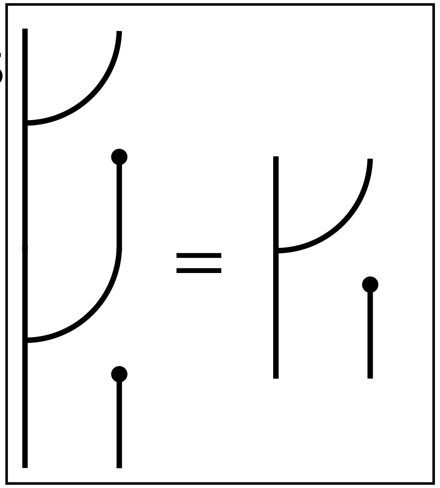
Recap Soergel Bimodules

- A direct summand of $\mathrm{BS}(w_1)(i_1) \oplus \cdots \oplus \mathrm{BS}(w_n)(i_n)$
- $\mathbb{S}\mathrm{Bim}$: objects as above; $\mathrm{Hom}(B, B') = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{R-\mathrm{gbim}}\big(B, B'(k)\big)$
- **Soergel categorification theorem**: $H \to [\mathbb{S}\mathrm{Bim}]_{\oplus}, b_s \mapsto [B_s]$ is a $\mathbb{Z}[v^{\pm 1}]$-algebra isomorphism (H is the associated Hecke algebra)
- Objects & morphisms hard to describe, algebraically/diagrammatically
- On the other hand, $\mathbb{S}\mathrm{Bim} = \mathrm{Kar}\,\mathbb{B}\mathbb{S}\mathrm{Bim}$, where

  $\mathrm{Kar}\,\mathcal{C}$ has objects $f \in \mathrm{Hom}(x, x)$, $f \circ f = f$ and morphisms

  $\varphi: f \to g,$

$$\begin{tikzcd} x \arrow[loop above, "f"] \arrow[rr, "\varphi"] & & y \arrow[loop above, "g"] \end{tikzcd} \qquad \varphi \circ f = \varphi = g \circ \varphi$$

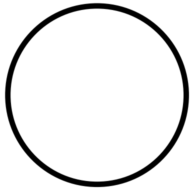


Goal: to present $\mathbb{B}\mathbb{S}\mathrm{Bim}$ diagrammatically

Recap Temperley–Lieb Category

- $\mathcal{TL}_\delta$ has objects $0,1,2,\ldots$ and morphisms crossingless matchings between m and n dots, e.g.

(no multivalent/univalent vertices)

subject to  $= \delta$ and isotopy

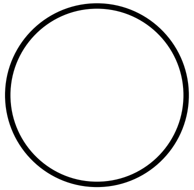
- **Temperley–Lieb algebras:** $\mathrm{TL}_{n,\delta} = \mathrm{Hom}_{\mathcal{TL}_\delta}(n,n)$. $\mathbb{Z}[\delta]$-algebras

Temperley-Lieb 2-Category

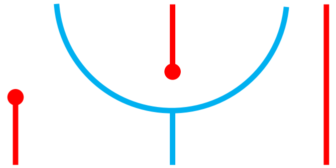
- $2\mathcal{TL}_\delta$: objects $=$ colored line segments (or);

  morphisms $=$ dots separating objects (e.g. );

  2-morphisms $=$ ($\mathbb{Z}[\delta]$-linear combinations of) crossingless matchings with alternatingly-colored regions, e.g.

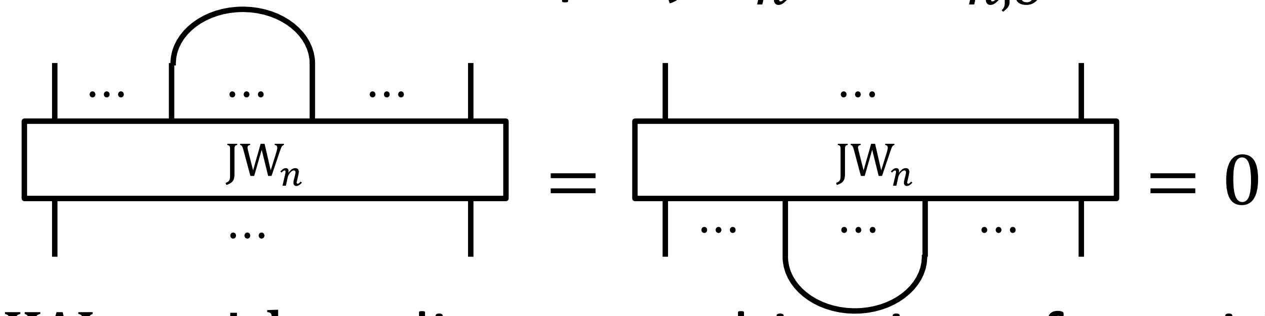
Relation:  $= \delta$

- Well-defined functor Σ: $2\mathcal{TL}_{\partial_s \alpha_t} \to \mathbb{BSB}\mathrm{im}$,  $\mapsto B_s B_t B_s B_t$,

 $\mapsto$  (deformation retract)

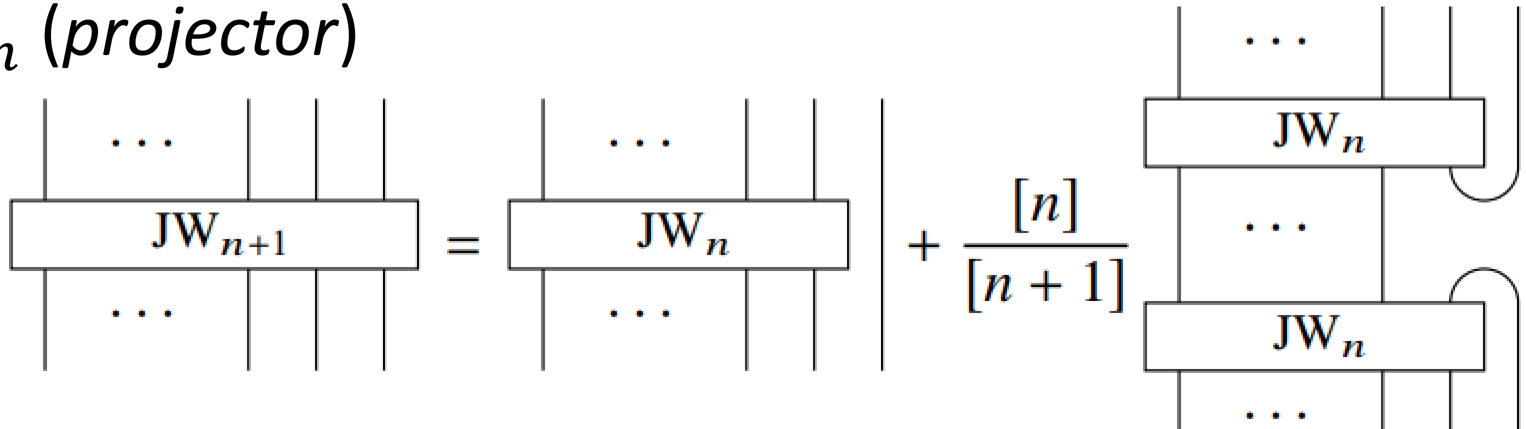
- Σ lifted to $\mathrm{Kar}\, 2\mathcal{TL}_{\partial_s \alpha_t}$ is fully faithful onto degree-0 maps.

Jones–Wenzl Projectors

- Tensor $\mathrm{TL}_{n,\delta}$ with $\mathbb{Q}(\delta) = \mathrm{Frac}\,\mathbb{Z}[\delta]$
- **Theorem.** There's unique $\mathrm{JW}_n \in \mathrm{TL}_{n,\delta}$, such that
  - 

  - $\mathrm{JW}_n = \mathrm{id}_n +$ linear combination of non-identity matchings

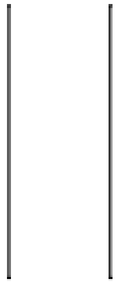
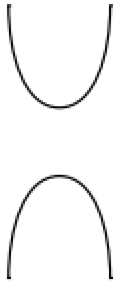
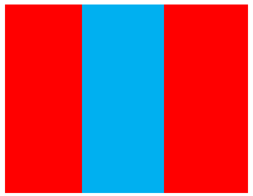
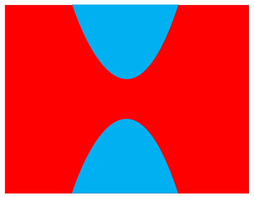
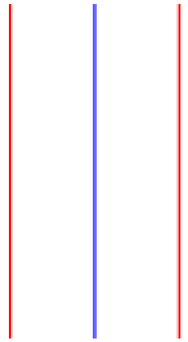
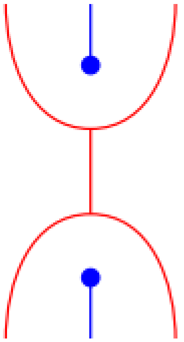
  Moreover, $\mathrm{JW}_n^2 = \mathrm{JW}_n$ (*projector*)
- **Recursive formula:**



$[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}; \delta = q + q^{-1}$

$[n]$ is a polynomial of δ

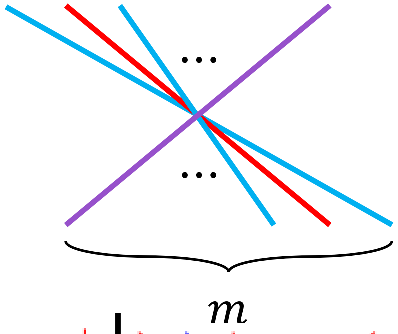
Jones–Wenzl Projectors (Cont’d)

- $\mathrm{JW}_1 =$; $\mathrm{JW}_2 =$  $- \frac{1}{\delta}$ 
- Escalate it to $2\mathcal{TL}_{2m_{st}}$: $\mathrm{JW}_2 =$  $- \frac{1}{2m_{st}}$ 
- Apply Σ: let $\mathrm{JW}_{(s,t,s)} \in \mathrm{Hom}_{\mathbb{BSBim}}(B_sB_tB_s, B_sB_tB_s)$ be  $- \frac{1}{\partial_s \alpha_t}$ 
- Similarly for $\mathrm{JW}_{(s,t,\ldots)}$

What’s $\mathrm{JW}_{(s,t)}$?

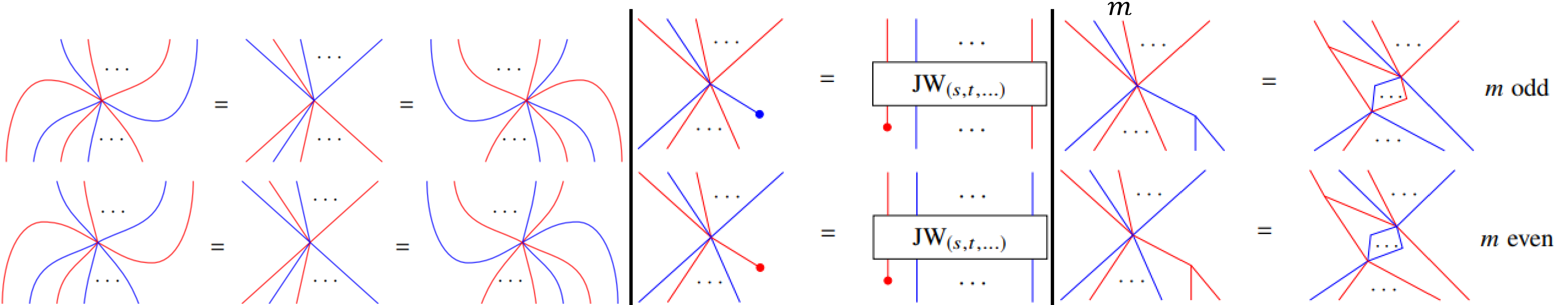
Recap Two-Color Soergel Calculus

- When $W = \overset{s}{\bullet}\overset{\infty}{\text{———}}\overset{t}{\bullet}$, $\mathcal{H}_{\mathrm{BS}}$ has objects $\underline{w}$'s and morphisms the same as those for the one-color calculus; $\mathcal{F}\colon \mathcal{H}_{\mathrm{BS}}(s) \to \mathbb{BSB}\mathrm{im}$ and $\mathcal{F}\colon \mathcal{H}_{\mathrm{BS}}(t) \to \mathbb{BSB}\mathrm{im}$ glue to fully-faithful $\mathcal{F}\colon \mathcal{H}_{\mathrm{BS}} \to \mathbb{BSB}\mathrm{im}$
- When $W = \overset{s}{\bullet}\overset{m}{\text{———}}\overset{t}{\bullet}$, additional morphisms



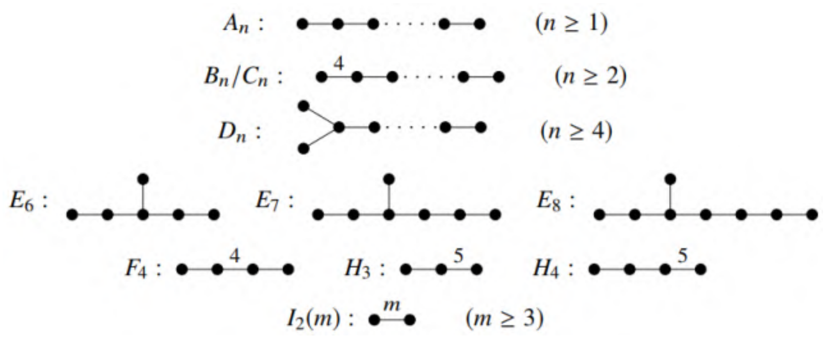
and additional relations

$m = 2$?

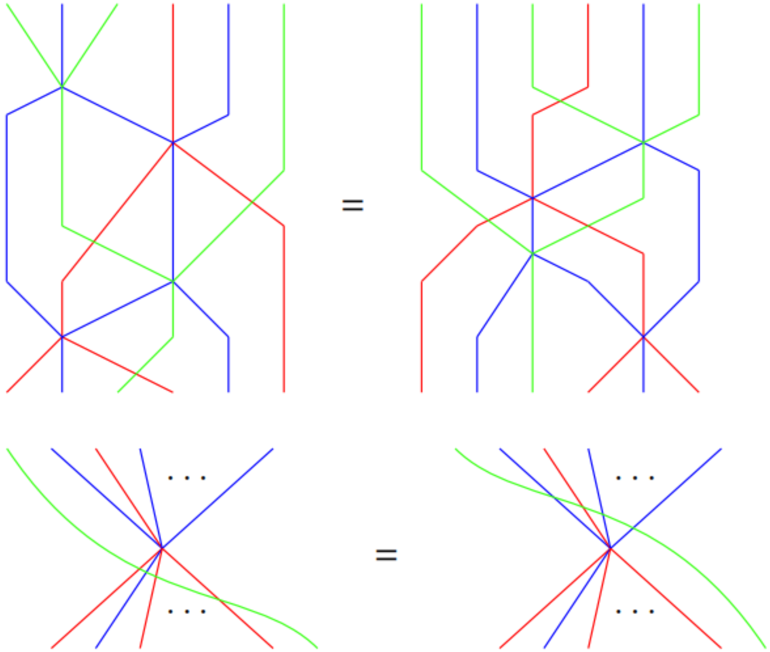


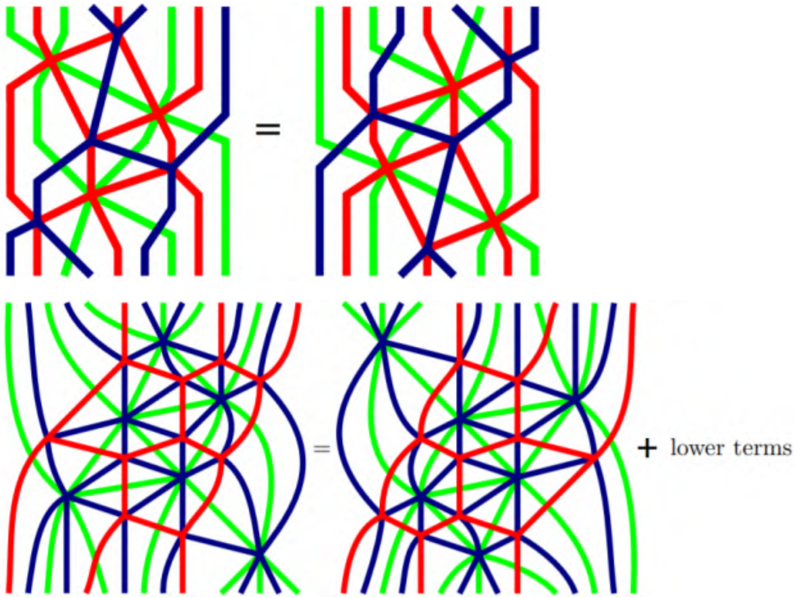
- Now an arbitrary (W, S), S finite
- $\mathcal{H}_{\mathrm{BS}}$: objects are $\underline{w}$’s; morphisms consist of
  - Univalent vertices
  - Trivalent vertices
  - $2m_{st}$-valent vertices
  - Boxes

and are subject to all 2-color relations plus the **3-color relations**

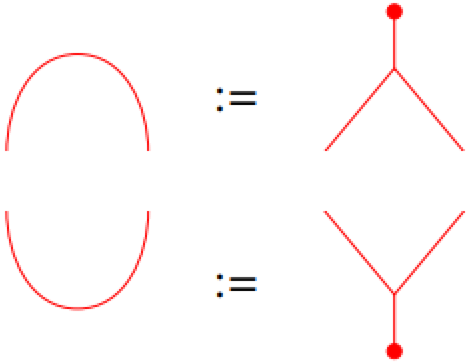


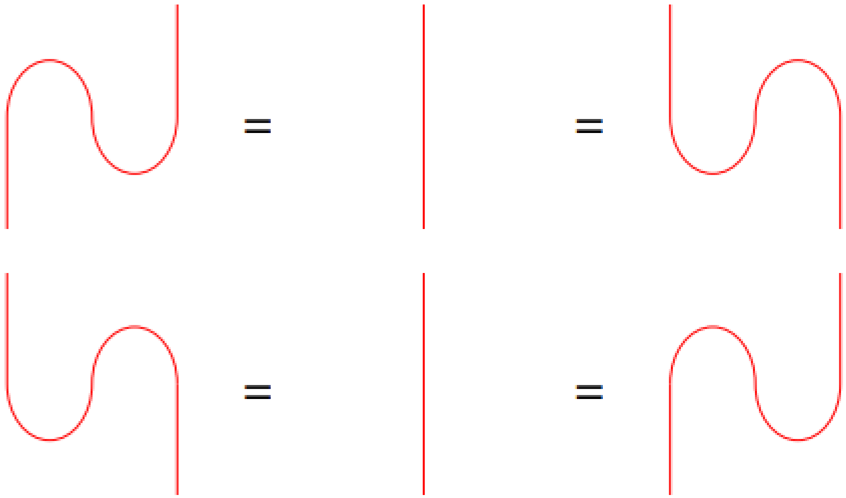
A_3, $A_1 \times I_2(m)$, B_3, H_3

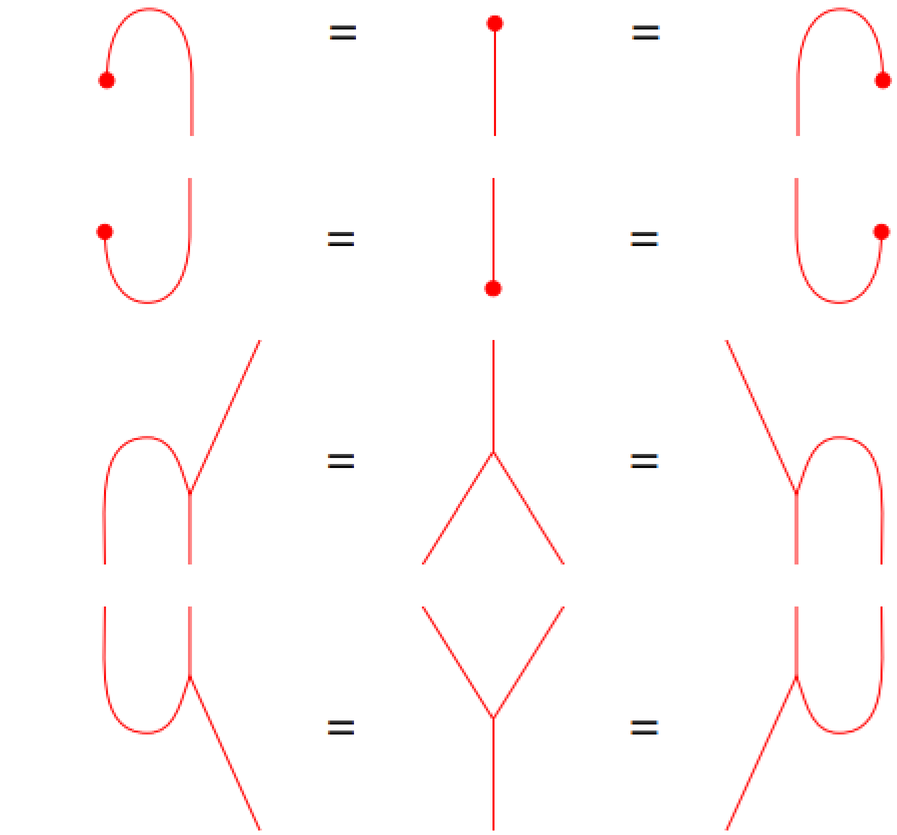


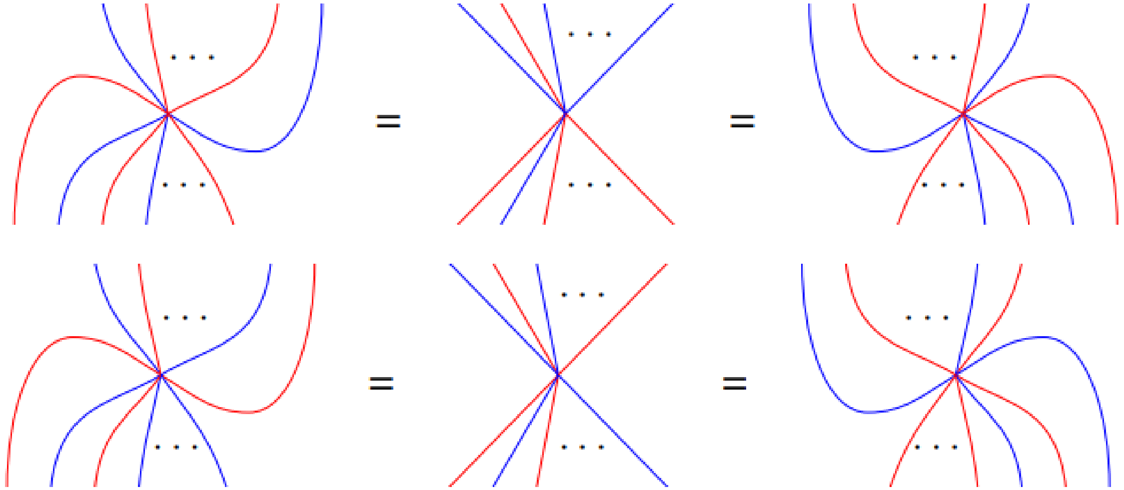


Isotopies Written Out





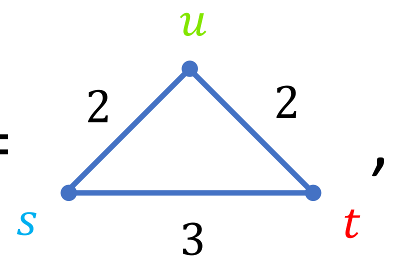




m odd

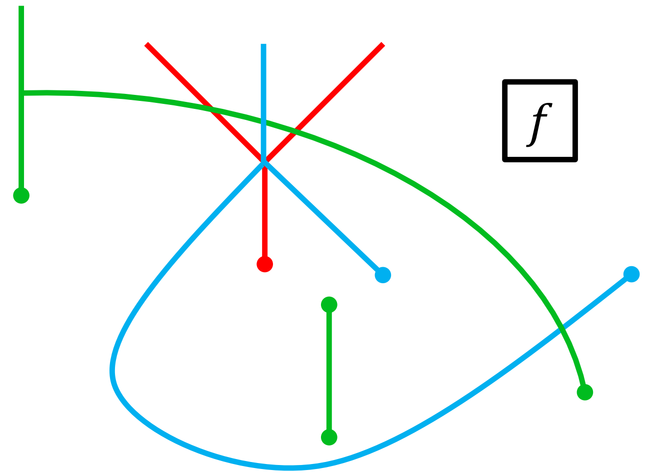
m even

• **Example.** $W =$

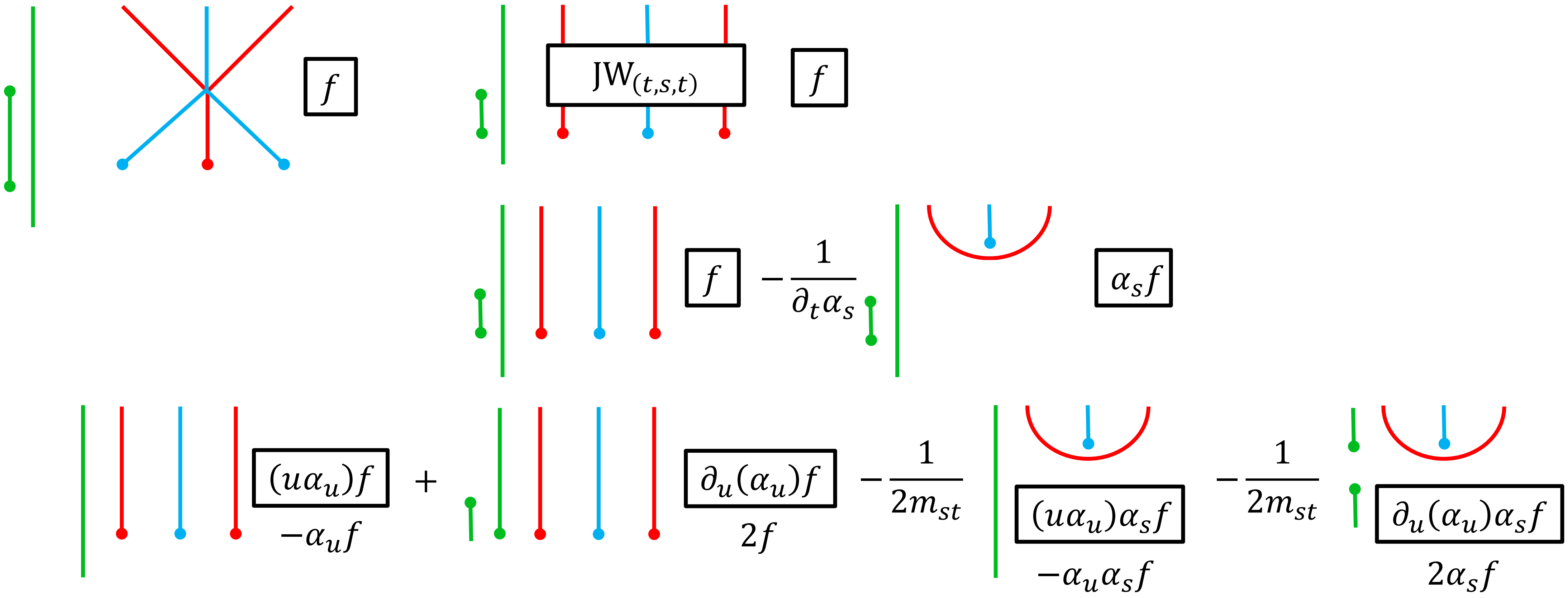


u
2
2
s
3
t

,

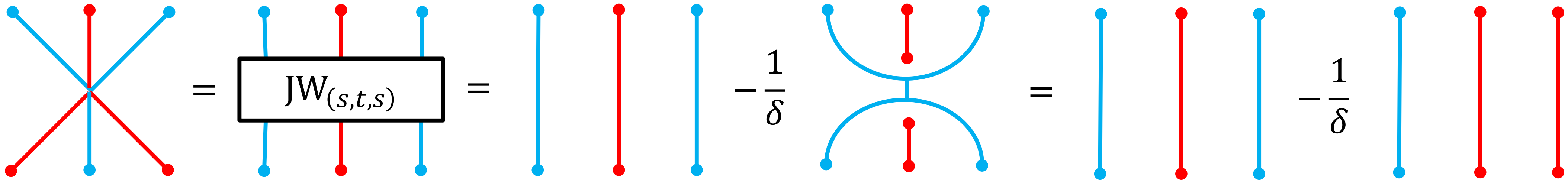


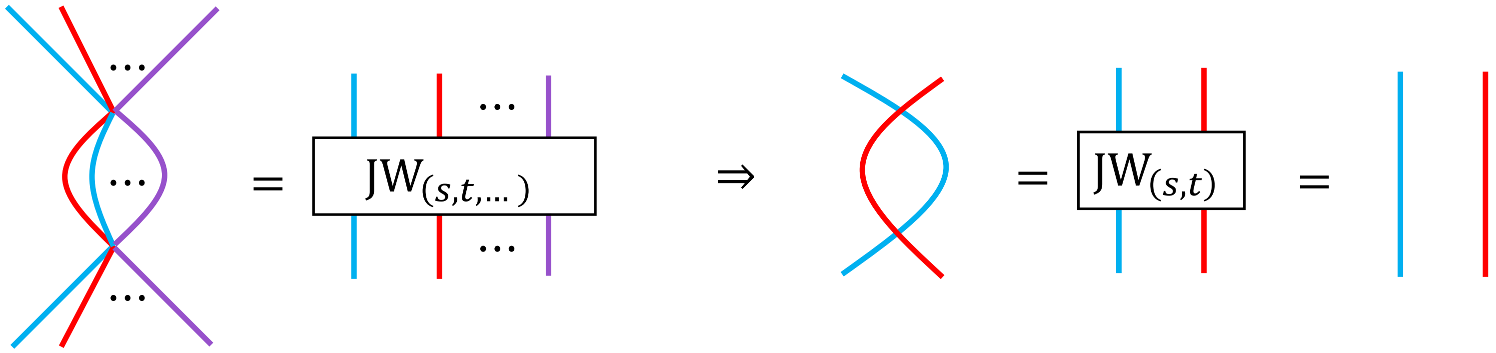
f



f
JW(t,s,t)
f
f
1
∂tαs
αsf
(uαu)f
−αuf
+
∂u(αu)f
2f
1
2mst
(uαu)αsf
−αuαsf
1
2mst
∂u(αu)αsf
2αsf

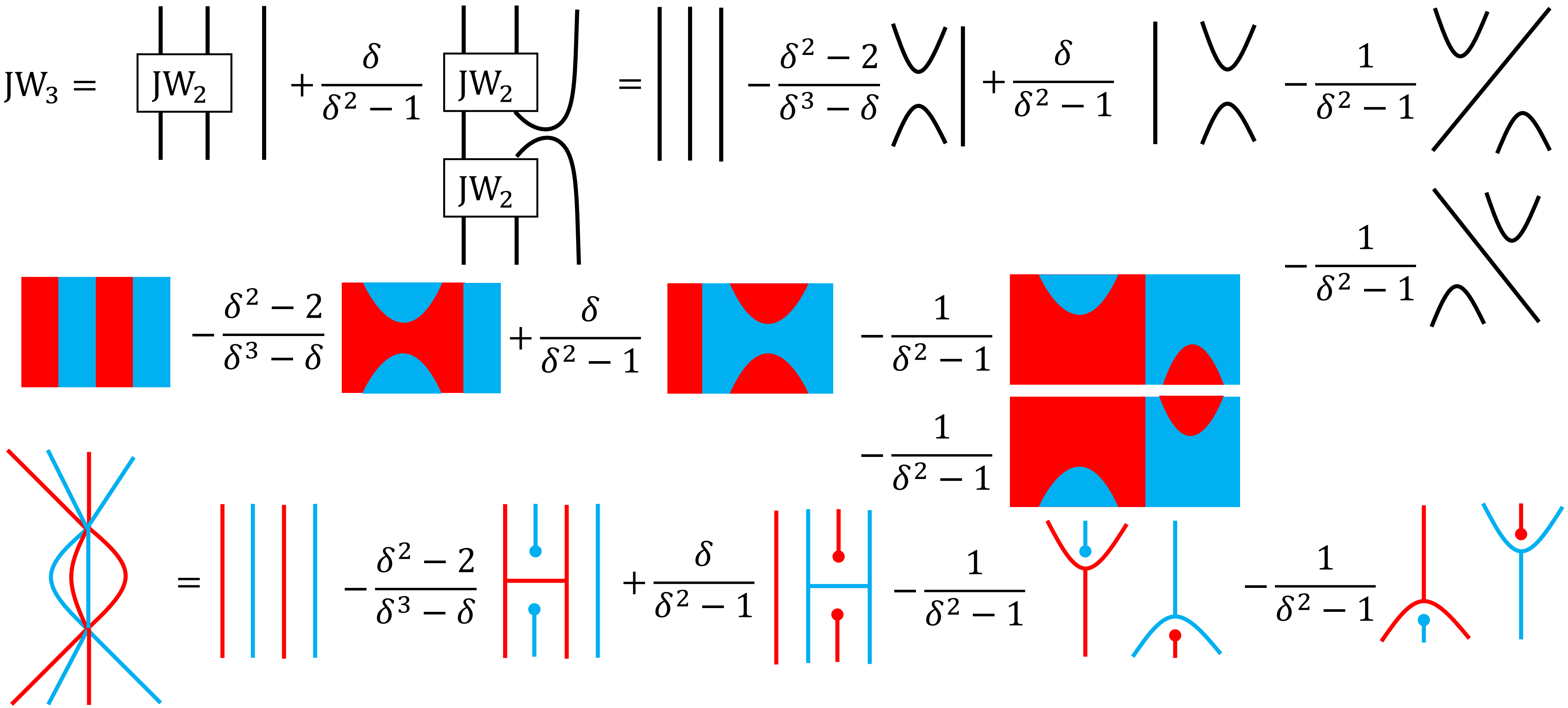
More Examples





Elias-Jones-Wenzl

More Examples



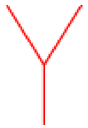
The Ultimate Categorical Equivalence

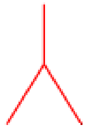
- $\mathcal{F}\colon \mathcal{H}_{\mathrm{BS}} \to \mathbb{BS}\mathrm{Bim}$, $s \mapsto B_s$, monoidal

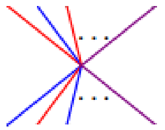
$\mathcal{F}\left(\boxed{f}\right)\colon \quad R \to R, \qquad 1 \mapsto f.$

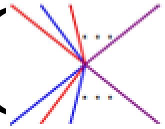
$\mathcal{F}\Big($  $\Big)\colon \quad B_s \to R, \qquad f \otimes g \mapsto fg.$

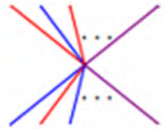
$\mathcal{F}\Big($  $\Big)\colon \quad R \to B_s, \qquad 1 \mapsto \frac{1}{2}(1 \otimes \alpha_s + \alpha_s \otimes 1).$

$\mathcal{F}\Big($  $\Big)\colon \quad B_s B_s \to B_s, \qquad 1 \otimes g \otimes 1 \mapsto \partial_s g \otimes 1.$

$\mathcal{F}\Big($  $\Big)\colon \quad B_s \to B_s B_s, \qquad f \otimes g \mapsto f \otimes 1 \otimes g.$

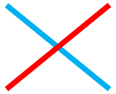
$\mathcal{F}\Big($  $\Big)\colon \quad \underbrace{B_s B_t \cdots}_{m_{st}} \to \underbrace{B_t B_s \cdots}_{m_{st}}, \qquad \underbrace{B_s B_t \cdots}_{m_{st}} \twoheadrightarrow B_{w_{s,t}} \hookrightarrow \underbrace{B_t B_s \cdots}_{m_{st}}$

$\mathcal{F}($  $)$

$\mathcal{F}($  $)$ $: \underbrace{B_sB_t\cdots}_{m_{st}} \to \underbrace{B_tB_s\cdots}_{m_{st}}, \qquad \underbrace{B_sB_t\cdots}_{m_{st}} \twoheadrightarrow B_{w_{s,t}} \hookrightarrow \underbrace{B_tB_s\cdots}_{m_{st}}$

- Let $w_{s,t}$ be the longest element of $\langle s,t\rangle$
- Then $B_sB_t\cdots$ and B_tB_s both contain $B_{w_{s,t}}$ as a direct summand (up to isomorphism) with multiplicity 1
- This gives $\twoheadrightarrow$ and $\hookrightarrow$. Rescale so that $1\otimes\cdots\otimes 1$ is sent to $1\otimes\cdots\otimes 1$
- **Example.** $W=$ 

$=$

$\mathcal{F}($  $): B_sB_t \twoheadrightarrow B_{st} \hookrightarrow B_tB_s$

$\xrightarrow{\sim}$

Walkthrough

- (W, S)
- H
- $\mathbb{BS}\mathrm{Bim}$
- $\mathbb{S}\mathrm{Bim} = \mathrm{Kar}\,\mathbb{BS}\mathrm{Bim}$
- $[\mathbb{S}\mathrm{Bim}]_{\oplus}$

$\cong$

Next Time

- Why there’re no 4-color relations
- Soergel categorification theorem