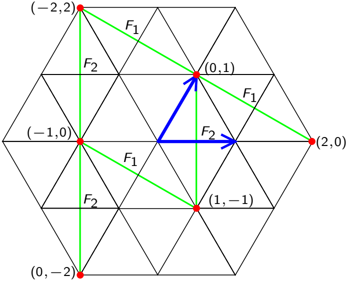
$\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory governs $\mathfrak{sl}_n$-link homology

Daniel Tubbenhauer

The m is not a typo!

May 2014

$= (F_2F_1F_2F_1 - q \cdot F_2F_2F_1F_1)v_{20}$



1. Motivation: The celebrated Jones polynomial
   - The Jones revolution
   - Its categorification

2. The quantum groups $\mathbf{U}_q(\mathfrak{sl}_m)$
   - A reminder on quantum weight-representations
   - The category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$

3. A diagrammatic presentation
   - $\mathfrak{sl}_2$-webs
   - Connection to $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$
   - How can one prove the graphical representation?

4. Connection to the $\mathfrak{sl}_n$-link polynomials
   - The Howe dual of Reshetikhin-Turaev
   - Links as F's
   - Categorification? Sure!

The famous Jones polynomial

Theorem(Jones 1984)

There is exactly one polynomial $J(\cdot)$ from the set of oriented link diagrams $\{L_D\}$ to $\mathbb{Z}[q, q^{-1}]$ with $J(\text{Unknot}) = 1$ that satisfies the skein relations

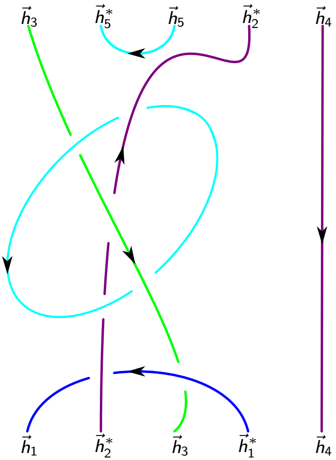
$$q^2 J(\nearrow\!\!\!\!\nwarrow) - q^{-2} J(\nwarrow\!\!\!\!\nearrow) = (q + q^{-1}) J(\uparrow\uparrow).$$

It is invariant under the three Reidemeister moves. Thus, it gives rise to a map from the set of all oriented links in S^3 to $\mathbb{Z}[q, q^{-1}]$: The Jones polynomial.

- Before Jones there was only one link polynomial: The Alexander polynomial.
- After Jones there where whole families of link polynomials.
- It was also extended to other set-ups.
- Nowadays the Jones polynomial is known to be related to different fields of modern mathematics and physics, e.g. the Witten-Reshetikhin-Turaev invariants of 3-manifolds originated from the Jones polynomial.
- Thus, we need to understand this better!

A tangle is an intertwiner

Let $\mathfrak{g}$ be any classical Lie algebra. Denote by $\vec{h}_i$ the $\mathbf{U}_q(\mathfrak{g})$-representation $V_{\vec{h}_i}$ of highest weight $\vec{h}_i$. Let T_D be a diagram of a, $\vec{h}_i$-colored, oriented tangle.



$\rightsquigarrow$

$$V_{\vec{h}_3} \otimes V_{\vec{h}_5^*} \otimes V_{\vec{h}_5} \otimes V_{\vec{h}_2^*} \otimes V_{\vec{h}_4}$$

$$\uparrow f(T_D)$$

$$V_{\vec{h}_1} \otimes V_{\vec{h}_2^*} \otimes V_{\vec{h}_3} \otimes V_{\vec{h}_1^*} \otimes V_{\vec{h}_4}$$

Representation theory does the trick!

Definition/Theorem(Reshetikhin-Turaev 1990)

Given the set-up from before we define a certain $\mathbf{U}_q(\mathfrak{g})$-intertwiner

$$f(T_D)\colon V_{\vec{h}_1}\otimes\cdots\otimes V_{\vec{h}_k}\to V_{\vec{h}_{k+1}}\otimes\cdots\otimes V_{\vec{h}_l}.$$

The $\mathbf{U}_q(\mathfrak{g})$-intertwiner $f(T_D)$ is an invariant of T_D.

In the case of colored, oriented links L_D we have

$$f(L_D)\colon\bar{\mathbb{Q}}\to\bar{\mathbb{Q}},\ 1\mapsto P_{\mathbf{RT}}(L_D)\in\mathbb{Z}[q,q^{-1}],$$

that is each configuration as above gives a polynomial invariant of oriented links! Restriction to $\mathfrak{sl}_2$ and the vector representation $\bar{\mathbb{Q}}^2$ gives the Jones polynomial.

Today: I will explain the "dual" of this.

Its categorification

Theorem(Khovanov 1999)

There is a chain complex $\mathbf{Kh}(\cdot)$ of graded vector spaces whose homotopy type is a link invariant. Its graded Euler characteristic gives the Jones polynomial.

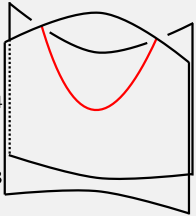
Theorem(Khovanov, Bar-Natan, Clark-Morrison-Walker,...)

The $\mathbf{Kh}(\cdot)$ can be extended to a functor from the category of links in S^3 to the category chain complexes of graded vector spaces.

$L \in S^3$

$C \in B^4$

$L' \in S^3$



$\overset{\mathbf{Kh}}{\longmapsto}$

gr. chain complex $\mathbf{Kh}(L)$

↑↓ gr. chain maps $\mathbf{Kh}(C)$

gr. chain complex $\mathbf{Kh}(L')$

History repeats itself

- Khovanov's construction can be extended to different set-ups.
- Rasmussen obtained from the homology an invariant that "knows" the slice genus and used it to give a combinatorial proof of the Milnor conjecture.
- Rasmussen also gives a way to combinatorial construct exotic $\mathbb{R}^4$.
- Kronheimer and Mrowka showed that Khovanov homology detects the unknot. This is still an open question for the Jones polynomial.
- Even better: Hedden-Ni and Batson-Seed proved that it detects unlinks. This is known to be false for the Jones polynomial.
- Before I forget: It is a strictly stronger invariant.

After Khovanov lots of other homologies of "Khovanov-type" were discovered. So we need to understand this better (I do not go into details today).

The quantum algebra $\mathbf{U}_q(\mathfrak{sl}_m)$

Definition

For $m \in \mathbb{N}_{>1}$ the quantum special linear algebra $\mathbf{U}_q(\mathfrak{sl}_m)$ is the associative, unital $\bar{\mathbb{Q}}(q)$-algebra generated by $K_i^{\pm 1}$ and E_i and F_i, for $i = 1, \ldots, m-1$ subject the following relations.

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q - q^{-1}},$$

$$K_iE_j = q^{(\epsilon_i,\alpha_j)}E_jK_i,$$

$$K_iF_j = q^{-(\epsilon_i,\alpha_j)}F_jK_i,$$

$$E_i^2E_j - [2]E_iE_jE_i + E_jE_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,$$

$$E_iE_j - E_jE_i = 0, \qquad \text{else},$$

$$F_i^2F_j - [2]F_iF_jF_i + F_jF_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,$$

$$F_iF_j - F_jF_i = 0, \qquad \text{else}.$$

Weyl beautiful theory of highest weights

Recall that a weight representation $V=\bigoplus_{\vec{k}\in\mathbb{Z}^m} V_{\vec{k}}$ of $\mathbf{U}_q(\mathfrak{sl}_m)$ is such that

$$V_{\vec{k}}=\{v\in V \mid K_i v = q^{(\vec{k}_i-\vec{k}_{i+1})}v\}.$$

Moreover, E_i, F_i jump around in the weight spaces, i.e.

$$E_i,F_i\cdot V_{\vec{k}}\subset V_{\vec{k}'},\quad \vec{k}'=\vec{k}\pm(\dots,\underbrace{1,-1}_{\text{pos.}i},\dots).$$

A vector $v_{\vec{h}}\in V$ is called highest weight vector of highest weight $\vec{h}$, if

$$E_i v_{\vec{h}}=0 \text{ for all } i \text{ and } v_h\in V_{\vec{h}} \text{ and } \mathbf{U}_q^-(\mathfrak{sl}_m)v_{\vec{h}}=V.$$

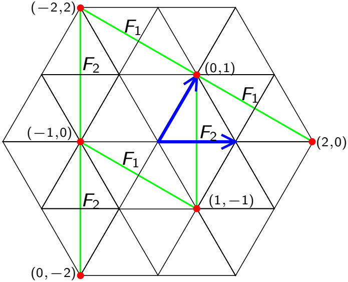
If $V_{\vec{h}}$ has a $v_{\vec{h}}$, then $V_{\vec{h}}$ is called highest weight representation. Magic:

Theorem(In finite dimensions!)

Two highest weight representations $V_{\vec{h}}, V_{\vec{h}'}$ are isomorphic iff $\vec{h}=\vec{h}'$. All $V_{\vec{h}}$ are irreducible and every irreducible $\mathbf{U}_q(\mathfrak{sl}_m)$-representation is isomorphic to a $V_{\vec{h}}$.

The weight lattice of $\mathfrak{sl}_m$ has rank $m-1$. Thus, to picture weight representations it is better to use $\vec{k} = (\vec{k}_1 - \vec{k}_2, \dots, \vec{k}_{m-1} - \vec{k}_m)$.

Then the $\mathbf{U}_q(\mathfrak{sl}_3)$-representation of highest weight $\vec{h} = (2,0,0) \mapsto \vec{h} = (2,0)$ is



The category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$

Definition

The representation category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of:

- The objects are finite tensor products of the $\mathbf{U}_q(\mathfrak{sl}_2)$-representations $\Lambda^k\bar{\mathbb{Q}}^2$. Denote them by $\vec{k} = (k_1, \dots, k_m)$ with $k_i \in \{0, 1, 2\}$.
- The 1-cells $w\colon \vec{k} \to \vec{k}'$ are $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.
- Composition of 1-cells is composition of intertwiners and $\otimes$ is the ordered tensor product.

Morally this category is enough: Every irreducible $\mathbf{U}_q(\mathfrak{sl}_2)$-representation $V_{\vec{h}}$ appears as a direct summand of an object of $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$. In fancier words:

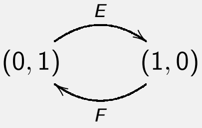
$$\mathbf{Kar}(\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))) \cong \mathbf{Rep}_{\mathrm{all}}(\mathbf{U}_q(\mathfrak{sl}_2)) \text{ (naturally).}$$

Exempli gratia

Example: $\mathbf{U}_q(\mathfrak{sl}_2)$-vector representation

Consider $\bar{\mathbb{Q}}^2$ with basis $x_{-1} = (0,1), x_{+1} = (1,0)$. These are called the weights -1 and $+1$ and K acts on them by $q^{\mp 1}$. The vector representation of $\mathbf{U}_q(\mathfrak{sl}_2)$ is:

Think: $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Think: $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

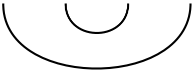
It is worth noting that $\Lambda^0\bar{\mathbb{Q}}^2 = \bar{\mathbb{Q}}$ is the trivial $\mathbf{U}_q(\mathfrak{sl}_2)$-representation, $\Lambda^2\bar{\mathbb{Q}}^2 \cong \bar{\mathbb{Q}}$ its dual and $\Lambda^1\bar{\mathbb{Q}}^2 = \bar{\mathbb{Q}}^2$ is the (self-dual) $\mathbf{U}_q(\mathfrak{sl}_2)$-vector representation above.

A $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiner is for example

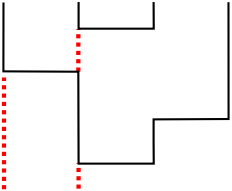
$$\mathrm{cup}\colon \underbrace{\Lambda^2\bar{\mathbb{Q}}^2 \otimes \Lambda^0\bar{\mathbb{Q}}^2}_{\cong\bar{\mathbb{Q}}} \to \underbrace{\Lambda^1\bar{\mathbb{Q}}^2 \otimes \Lambda^1\bar{\mathbb{Q}}^2}_{\cong\bar{\mathbb{Q}}^2\otimes\bar{\mathbb{Q}}^2},\ \ 1 \mapsto x_{+1} \otimes x_{-1} - q^{-1} \cdot x_{-1} \otimes x_{+1}.$$

Think topological but write algebraical

Think:



Write:



Advantage: Decomposition à la Morse into basic pieces.

Ignore dotted red lines: We used them to solve sign issues (functoriality of Khovanov homology for example). They encode the fact for quantum groups the antipode (dual representations) comes with a sign.

The (rigid) $\mathfrak{sl}_2$-webs - the objects

Definition - Part I

The (rigid) $\mathfrak{sl}_2$-web spider $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of the following.

The objects are m-tuples

$$\vec{k} = (k_1, \dots, k_m) \quad \text{such that} \quad \sum_{j=1}^{m} k_j = d, k_j \in \{0, 1, 2\}.$$

Example:

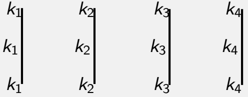
$$d = 10: \quad \vec{k}_1 = (2, 2, 0, 1, 2, 0, 1, 2, 0, 0) \quad \text{and} \quad \vec{k}_2 = (2, 2, 2, 2, 2, 0, 0, 0, 0, 0)$$

The (rigid) $\mathfrak{sl}_2$-webs - the generating 1-morphisms

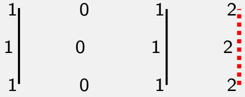
Definition - Part II

The generating 1-cells are $w\colon \vec{k} \to \vec{k}'$ are edge-labeled graphs with labels from the set $\{0,1,2\}$ (We do not draw 0-edges and 2-edges dotted) such that

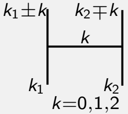
- The generators are either identities



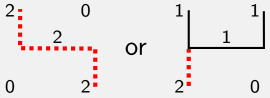
For example:



- Or ladders



For example:

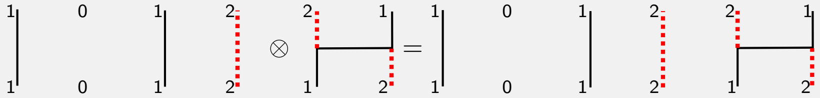


- All 1-cells should be generated by identities and ladders by $\circ$ and $\otimes$, where the $\bar{\mathbb{Q}}(q)$-linear composition $\circ$ is stacking (see next page).

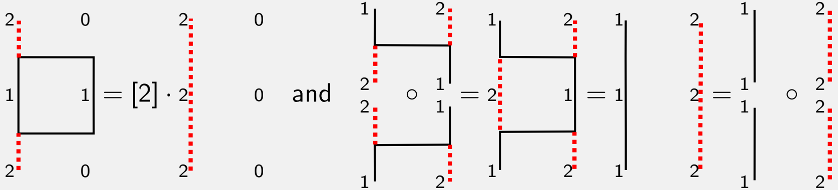
The (rigid) $\mathfrak{sl}_2$-webs - and all the rest

Definition - Part III

- The monoidal structure $\otimes$ is given by juxtaposition, e.g.



- Relations are the circle removals and isotopies, e.g. $([2] = q + q^{-1})$



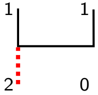
Intertwiner are pictures

Theorem(Kuperberg 1997, $n > 3$: Cautis-Kamnitzer-Morrison 2012)

The 1-categories $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ and $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ are equivalent.

Example: cup=cup, i.e.

$$\mathrm{cup}\colon \Lambda^2\bar{\mathbb{Q}}^2 \otimes \Lambda^0\bar{\mathbb{Q}}^2 \to \Lambda^1\bar{\mathbb{Q}}^2 \otimes \Lambda^1\bar{\mathbb{Q}}^2 \quad \mapsto$$



Question

How can one prove such a statement?

Finding the generators for $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ is doable, but...

Finding a complete set of relations is very hard!

The idempotented version

Definition(Beilinson-Lusztig-MacPherson)

For each $\vec{k} \in \mathbb{Z}^{m-1}$ adjoin an idempotent $1_{\vec{k}}$ (think: projection to the $\vec{k}$-weight space!) to $\mathbf{U}_q(\mathfrak{sl}_m)$ and add some relations, e.g.

$$1_{\vec{k}}1_{\vec{k}'} = \delta_{\vec{k},\vec{k}'}1_{\vec{k}} \text{ and } K_{\pm i}1_{\vec{k}} = q^{\pm \vec{k}_i}1_{\vec{k}} \text{ (no } K's \text{ anymore!)}.$$

and the E's and F's still jump around, e.g.

$$1_{\vec{k}-\overline{\alpha}_i}F_i1_{\vec{k}} = F_i1_{\vec{k}} = 1_{\vec{k}-\overline{\alpha}_i}F_i.$$

The idempotented quantum special linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_m) = \bigoplus_{\vec{k},\vec{k}' \in \mathbb{Z}^{m-1}} 1_{\vec{k}}\mathbf{U}_q(\mathfrak{sl}_m)1_{\vec{k}'}.$$

An important fact: The $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ has the "same" representation theory as $\mathbf{U}_q(\mathfrak{sl}_m)$.

An instance of q-skew Howe duality

The commuting actions of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ on

$$\bigoplus_{a_1+\cdots+a_m=d} (\Lambda^{a_1}\bar{\mathbb{Q}}^2 \otimes \cdots \otimes \Lambda^{a_m}\bar{\mathbb{Q}}^2) \cong \Lambda^d(\bar{\mathbb{Q}}^m \otimes \bar{\mathbb{Q}}^2) \cong \bigoplus_{a_1+a_2=d} (\Lambda^{a_1}\bar{\mathbb{Q}}^m \otimes \Lambda^{a_2}\bar{\mathbb{Q}}^m)$$

introduce a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action on the left side and a $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$-action on the right side.

The left and right side are $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$- and $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$-weight spaces with weights

$$\vec{k}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_m)} = (a_1 - a_2, \ldots, a_{m-1} - a_m) \quad \text{and} \quad \vec{k}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_2)} = (a_1 - a_2).$$

Here the $\Lambda^k\bar{\mathbb{Q}}^l_q$ are irreducible $\dot{\mathbf{U}}_q(\mathfrak{sl}_l)$-representations ($l \in \{2, m\}$).

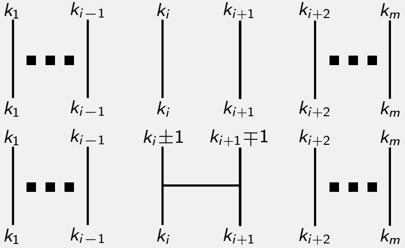
Graphical quantum skew Howe duality

Theorem

There is an $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action on $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))^m$ (objects of length m)!

$1_{\vec{k}} \mapsto$

$E_i 1_{\vec{k}},\ F_i 1_{\vec{k}} \mapsto$



That is, we stack these pictures on top of a given $\mathfrak{sl}_2$-web.

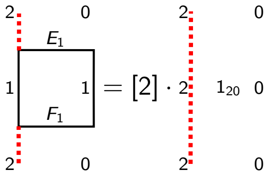
Thus, $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))^m$ is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module and not just a $\mathbf{U}_q(\mathfrak{sl}_2)$-module.

An instance of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory

What is the upshot of this?

- "Explains" the $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiner as instances of the (well-developed) $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory.
- The action of the F's is explicit and inductive - a powerful tool to prove statements.
- All the relations follow from the well-known ones from $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$, e.g.

$$E_1F_1v_{20} - \underbrace{F_1E_1v_{20}}_{=0} = \underbrace{\frac{K_1K_2^{-1} - K_1^{-1}K_2}{q - q^{-1}}}_{=[2]1_{20} \text{ in } \dot{\mathbf{U}}_q(\mathfrak{sl}_m)} v_{20} \Rightarrow$$



- Even better: $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ suffices for everything!

Let L_D be a diagram of an oriented link. Set $[2] = q + q^{-1}$ and

n_+ = number of crossings  n_- = number of crossings 

Definition/Theorem(Jones 1984, Kauffman 1987)

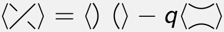
The bracket polynomial of the diagram L_D (without orientations) is a polynomial $\langle L_D \rangle \in \mathbb{Z}[q, q^{-1}]$ given by the following rules.

- $\langle \emptyset \rangle = 1$ (normalization).
- $\langle$  $\rangle = \langle$  $\rangle - q\langle$  $\rangle$ (recursion step 1).
- $\langle$  $\amalg L_D \rangle = [2] \cdot \langle L_D \rangle$ (recursion step 2).
- $[2]J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$ (Re-normalization).

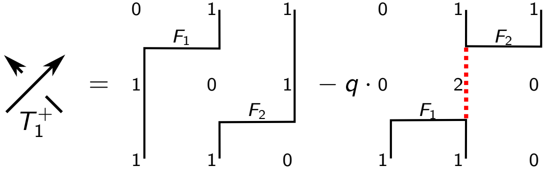
The polynomial $J(\cdot) \in \mathbb{Z}[q, q^{-1}]$ is an invariant of oriented links.

Observation(Reshetikhin-Turaev 1990)

We can read the right side of



as certain $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.



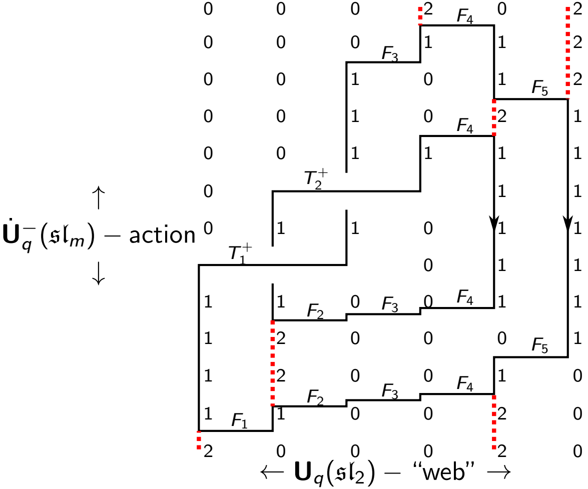
$\overset{\text{Howe}}{\underset{\text{dual}}{=}} F_1F_2v_{110} - q \cdot F_2F_1v_{110}.$

Note: It is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight module: No E's are needed!

Exercise: Do the negative .

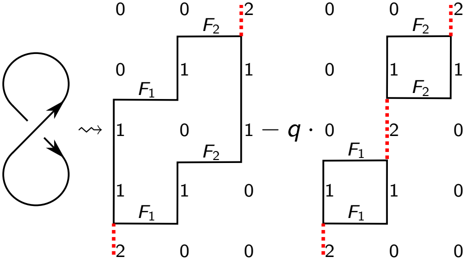
$\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ knows link diagrams

Using these T_k^+ and T_k^- together with the F's we can write link diagrams as



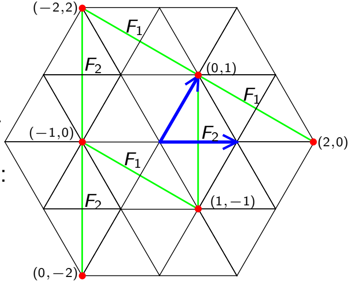
$$\mathrm{qH}(\mathbf{Hopf}) = F_4^{(2)} F_4 F_3 F_5 F_4 T_2^+ T_1^+ F_4 F_3 F_2 F_5 F_4 F_3 F_2 F_1 F_4^{(2)} F_3^{(2)} F_2^{(2)} v_{220000}.$$

Jumping from a highest to a lowest weight



$\overset{\text{Howe}}{\underset{\text{dual}}{=}}$ $F_2 F_1 F_2 F_1 v_{200} - q \cdot F_2 F_2 F_1 F_1 v_{200}$

Or in terms of weight diagrams:

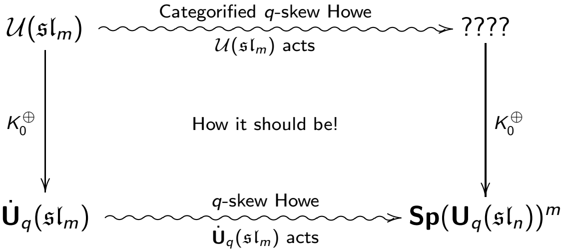


The $\mathfrak{sl}_n$-link polynomials using $\mathfrak{sl}_m$-symmetries

Let us summarize the connection between (colored) $\mathfrak{sl}_n$-link polynomials and the $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-$\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-skew Howe duality.

- Reshetikhin-Turaev: The $\mathfrak{sl}_n$-link polynomials $P^n_{\mathbf{RT}}(\cdot)$ are $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiner.
- $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiner are vectors in hom's between $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight spaces.
- Only F's: $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ suffices. Conclusion: The (colored) $\mathfrak{sl}_n$-link polynomials $P^n_{\mathbf{RT}}(\cdot)$ are instances of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory!
- Even better: There exists a fixed m for each link L such that $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory governs all the $\mathfrak{sl}_n$-polynomials of L.
- If L_D is a link diagram, then $P^n_{\mathbf{RT}}(L_D)$ is obtained by jumping via F's from a highest $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight v_h to a lowest $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight v_l!
- Guess: Should work in the types B, C, D as well.

The overview



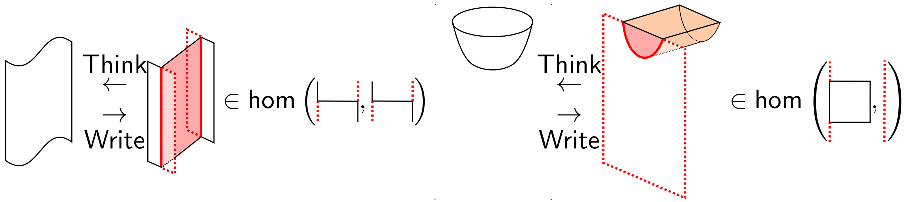
This is how it should be: There is an $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action on the $\mathfrak{sl}_n$-web spiders (for us it was mostly the case $n = 2$)

On the left side: There is Khovanov-Lauda's categorification $\mathcal{U}(\mathfrak{sl}_m)$ of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$.

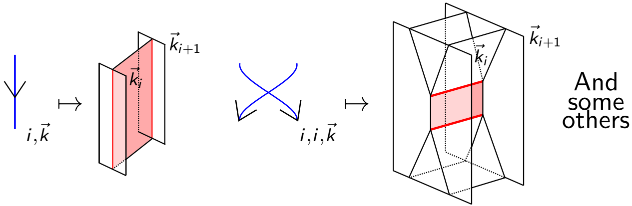
Conclusion: There should be a 2-action of $\mathcal{U}(\mathfrak{sl}_m)$ on the top right - a suitable 2-category of "natural transformations" between $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners!

And it works: Categorified q-skew Howe duality

The top left is the (rigid) $\mathfrak{sl}_2$-foam 2-category **Foam$_2$**.



On 2-**cells:** We define an 2-action



And play the same story again on a "higher" level...

The $\mathfrak{sl}_n$-homologies using $\mathfrak{sl}_m$-symmetries

Let us summarize the connection between $\mathfrak{sl}_n$-homologies and the higher q-skew Howe duality.

- Khovanov, Khovanov-Rozansky and others: The $\mathfrak{sl}_n$-link homology can be obtained using certain "$\mathfrak{sl}_n$-foams".
- Only F's: The (cyclotomic) KL-R suffices.
- Conclusion: The $\mathfrak{sl}_n$-link homologies are instances of highest $\mathcal{U}(\mathfrak{sl}_m)$-weight representation theory!
- Or in short: It is the usual "higher representation theory Yoga", aka replace weight spaces by weight categories, actions by functors and add the natural transformations.
- Guess: Should work in the types B, C, D as well.
- Guess: Should be honestly computable.
- Guess: The m is fixed! Stabilizing effects?

There is still much to do...

Thanks for your attention!