Quivers with Relations

Seminar: Representation Theory of Algebras FS2020

Simon Egli
Yu Lucy Huan

0. Recall of definitions

We first recall the def. of quivers.

DEF: A quiver $Q = (Q_0, Q_1, s, t)$ consists of:
Q_0 a set of vertices
Q_1 a set of arrows
$s: Q_1 \to Q_0$ a map from arrows to vertices, mapping an arrow to its starting point
$t: Q_1 \to Q_0$ a map from arrows to vertices, mapping an arrow to its terminal point.

We represent an element $\alpha \in Q_1$ by drawing an arrow from its starting point $s(\alpha)$ to its endpoint $t(\alpha)$ as follows:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha)$$

DEF: Let Q be a quiver. The path algebra kQ of Q is the algebra with basis the set of all paths in the quiver Q and with multiplication def. on two basis elements c, c' by

$$cc' = \begin{cases} c \cdot c' & \text{if } s(c') = t(c) \\ 0 & \text{otherwise} \end{cases}$$

Thus the product of two arbitrary elements $\sum_c \lambda_c c$, $\sum_{c'} \lambda'_{c'} c'$ of kQ is given by $\sum_{cc'} \lambda_c \lambda_{c'} cc'$.

DEF: A path of the form
$$\begin{tikzcd} i \arrow[r, "\alpha_1"] & \bullet \arrow[r, "\alpha_2"] & \bullet \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{l-1}"] & \bullet \arrow[llll, bend left, "\alpha_l"] \end{tikzcd}$$
given by $(i \mid \alpha_1, \alpha_2, \ldots, \alpha_l \mid i)$ is called an oriented cycle.

1. ADMISSIBLE IDEALS AND QUOTIENTS OF PATH ALGEBRAS

Let $Q = (Q_0, Q_1)$ be a finite quiver, and let $A = kQ$ be its path algebra. If the quiver Q has oriented cycles, then the path algebra is infite-dimensional. We want to consider quotients of the path algebra by certain ideals and we would like these quotients to be finite dimensional and indecomposable as an algebra $\Rightarrow$ concept of admissible ideals.

DEF: The arrow ideal R_Q of A is the two-sided ideal generated by all arrows in Q.
As a vector space, we can decompose the arrow ideal as $R_Q = kQ_1 \oplus kQ_2 \oplus \dots \oplus kQ_\ell \oplus \dots$
where kQ_ℓ is the subspace of kQ with basis the set Q_ℓ of paths of length ℓ.

The ℓ-th power of the arrow ideal can be decomposed as $R_Q^\ell = \bigoplus_{m \geq \ell} kQ_m$, and it has a basis consisting of all paths of length greater or equal to ℓ.

DEF: A two-sided ideal I of kQ is called an admissible ideal if there exists an integer $m \geq 2$ s.t. $R_Q^m \subseteq I \subseteq R_Q^2$.

If I is an admissible ideal of kQ, then (Q, I) is called a bound quiver and the quotient algebra kQ/I is called a bound quiver algebra.

Remark 1: We have that $R_Q^m \subseteq I \Rightarrow$ the admissible ideal I contains all paths of length greater or equal to m, which guarantees that the bound quiver algebra is finite-dimensional.

If the quiver does not contain any oriented cycles, then there always exists m s.t. $R_Q^m \subseteq I$. It suffices to take m greather than the length of the longest path in Q.
Thus if Q has no oriented cycles, we have an ideal is admissible $\Leftrightarrow$ it is contained in R_Q^2

The condition $I \subseteq R_Q^2$ guarantees that we do not cut any arrows when we take the quotient; thus the bound quiver algebra is connected.

Remark 2: Suppose that I is an admissible ideal which is generated by the elements $\sigma_1, \sigma_2, \dots, \sigma_s$.
For every pair of vertices x, y, the element $e_x \sigma_i e_y$ is a linear combination of paths from x to y, hence a relation.
Since $\sigma_i = \sum_{x,y} e_x \sigma_i e_y$, we see that the ideal I is also generated by the set of relations $\{e_x \sigma_i e_y \mid i = 1, 2, \dots, s;\ x, y \in Q_0\}$. This shows that for every admissible ideal I, $\exists$ a set of relations that generate I.

Remark 3: A finite-dimensional algebra A is called basic if for every set of primitive orthogonal idempotents $\{e_1, \dots, e_n\}$ s.t. $1 = e_1 + \dots + e_n$, we have $e_i A \cong e_j A \Leftrightarrow i = j$

If A is <u>not</u> basic, define $e_A = e_{s_1} + \dots + e_{s_t}$ to be the sum of a maximal set of primitive orthogonal idempotents s.t. $e_{s_i} A \cong e_{s_j} A \Leftrightarrow i = j$. Then it can be shown:
$e_A A e_A$ is basic and the module categories of A and $e_A A e_A$ are equivalent.
Hence, from the point of view of representation theory, it suffices to consider only

basic algebras.

It can also be shown that every basic finite-dimensional k-algebra is isomorphic to a quotient of a path algebra by an admissible ideal. The vertices of the quiver of that path algebra are in bijection with a set of primitive, orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$ with the property $1 = e_1 + \dots + e_n$ and the number of arrows from e_i to e_j is the dimension of the vector space $e_i(\operatorname{rad} A/(\operatorname{rad} A^2))e_j$

We can rephrase the above remark as follows:

> From the point of view of representation theory, the study of finite-dimensional k-algebras reduces to the study of bound quiver algebras.

We now give some examples regarding admissible ideals.

EXAMPLE 1. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \end{tikzcd}$$

Then the ideal $I = \langle \alpha^2\beta, \alpha^3 \rangle$ is admissible: take $m=3$, then any path of length ≥ 3 must contain α^3 or $\alpha^2\beta$ as a subpath. $\Rightarrow R_Q^3 \subseteq I$. $I \subseteq R_Q^2$ is clear since the generators of I are of length 3.

Now we give an example that shows different relations on the same quiver may lead to isomorphic algebras.

EXAMPLE 2. Let Q be the quiver

$$\begin{tikzcd} & 2 \arrow[dl, "\beta"'] & \\ 1 & & 4 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & 3 \arrow[ul, "\delta"] & \end{tikzcd}$$

and def. two admissible ideals $I_1 = \langle \alpha\beta + \gamma\delta \rangle$ and $I_2 = \langle \alpha\beta - \gamma\delta \rangle$. Then $I_1 \neq I_2$ if $\operatorname{char}(k) \neq 2$, but the corresponding bound quiver algebras are isomorphic: $kQ/I_1 \cong kQ/I_2$

Chapter 2

So far we studied two categories: the category of finite-dimensional quiver representations rep Q and the category of finite generated kQ-modules mod kQ. Now we will show that these two are equivalent. That means that modules and representations are essentially the same. This is quite useful, f.e. when computing some quivers, it's more convenient to use the representations with the graphical intuition.

Thm 4. Let $A = kQ/I$, where Q is a finite connected quiver and I is an admissible ideal. Then there is an equivalence of categories between the category mod$A$ of finitely generated right A-modules and the category rep(Q,I) of finite-dimensional bound quiver representations:

$$\text{mod}A \cong \text{rep}(Q,I)$$

Outline of the proof:

We give just an outline of the proof, since it is quite long and technical. We construct two functors $F: \text{mod}A \to \text{rep}(Q,I)$ and $G: \text{rep}(Q,I) \to \text{mod}A$, s.t. $F \circ G \cong 1_{\text{rep}(Q,I)}$ and $G \circ F \cong 1_{\text{mod}A}$. We just have to define them, and show that they are well defined and finally that they are functors.

Remark: From that theorem we can follow that many of the previous results on quivers representations, which we have proofed, hold also for bound quiver representations and modules.

Def. 2 An algebra A is called hereditary if each submodule of a projective module is projective.

Prop. 6. Path algebras of quivers without oriented cycles are hereditary.

3. PROJECTIVE REPRESENTATION OF BOUND QUIVERS

Let (Q, I) be a bound quiver and $A = kQ/I$ its bound quiver algebra. For every vertex $i \in Q_0$, we will def. an indecomposable projective representation $P(i)$ and an indecomposable injective representation $I(i)$ of the bound quiver (Q, I).

DEF: Let i be any vertex in Q.

(a) $P(i) = (P(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$, where $P(i)_j$ is the k-vector space with basis the set of all residue classes $c + I$ of paths c from i to j in Q;
and if $j \xrightarrow{\alpha} \ell$ is an arrow in Q, then $\varphi_\alpha : P(i)_j \longrightarrow P(i)_\ell$ is the linear map def. on the basis by composing the paths from i to j with the arrow $j \xrightarrow{\alpha} \ell$, that is
$\varphi_\alpha(c + I) = c\alpha + I$

(b) $I(i) = (I(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$ where $I(i)_j$ is the k-vector space with basis the set of all residue classes $c + I$ of paths c from j to i in Q;
and if $j \xrightarrow{\alpha} \ell$ is an arrow in Q, then $\varphi_\alpha : I(i)_j \longrightarrow I(i)_\ell$ is the linear map def. on the basis by deleting the arrow $j \xrightarrow{\alpha} \ell$ from those paths from j to i which start with α and sending to zero the paths that do not start with α, that is,

$$\varphi_\alpha(c + I) = \begin{cases} c' + I & \text{if } c = \alpha c' \\ 0 & \text{otherwise} \end{cases}$$

We now give an example:

EXAMPLE: Let Q be the quiver

$$\begin{tikzcd}[row sep=small] & & & 2 \arrow[dl, "\beta"'] & \\ 6 & 5 \arrow[l, "\sigma"'] & 4 \arrow[l, "\rho"'] & & 1 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & & & 3 \arrow[ul, "\delta"] & \end{tikzcd}$$

and $I = \langle \alpha\beta - \gamma\delta, \beta\rho, \delta\rho\sigma \rangle$. Then

$P(1) = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \end{smallmatrix}$ $\quad P(2) = \begin{smallmatrix} 2 \\ 4 \end{smallmatrix}$ $\quad P(3) = \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}$ $\quad P(4) = \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix}$ $\quad P(5) = \begin{smallmatrix} 5 \\ 6 \end{smallmatrix}$ $\quad P(6) = 6$

DEF: (a) Let A be a finite-dimensional k-algebra, and let $M \in \operatorname{mod} A$.
The intersection of all maximal submodules of M is called the radical $\operatorname{rad}(M)$ of the module M.
(b) The quotient $M/\operatorname{rad} M$ is called the top of M and its denoted by $\operatorname{top} M$.

LEM: Let $A = kQ/I$ be a bound quiver algebra and $P(i) = (P(i)_j, \varphi_\alpha)$ the indecomposable projective representation at vertex i. Then $\operatorname{rad} P(i) = (P(i)'_j, \varphi'_\alpha)$, where $P(i)'_j = P(i)_j$ if $i \neq j$, and $P'(i)_i$ is the vector space spanned by $\{c + I \mid c \text{ is a nonconstant path from } i \text{ to } i\}$ and
$\varphi'_\alpha = \varphi_\alpha|_{P(i)'_{s(\alpha)}}$.

In particular, $\operatorname{top}(P) \cong S(i)$.

Proof will be omitted.

EXAMPLE (CONTINUE): In the example above, we have

$$\operatorname{rad} P(1) = \begin{smallmatrix} 2 & 3 \\ 4 & \end{smallmatrix} \qquad \operatorname{rad} P(2) = 4 \qquad \operatorname{rad} P(3) = \begin{smallmatrix} 4 \\ 5 \end{smallmatrix}$$

$$\operatorname{rad} P(4) = \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \qquad \operatorname{rad} P(5) = 6 \qquad \operatorname{rad} P(6) = 0$$

4. HOMOLOGICAL DIMENSION

In this section, we def. the projective dimension ("measures how far a module is from being projective") and the global dimension ("measures how far an algebra is from being hereditary") of an algebra.

DEF: Let M be an A-module. The projective dimension $\operatorname{pd} M$ of M is the smallest integer d s.t. there exists a projective resolution of the form

$$0 \longrightarrow P_d \longrightarrow P_{d-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

If no such resolution exists, then we say that M has infinite projective dimension.

Dually, the injective dimension $\operatorname{id} M$ of M is the smallest integer d s.t. there exists an injective resolution of the form

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \dots \longrightarrow I_{d-1} \longrightarrow I_d \longrightarrow 0.$$

If no such resolution exists, then we say that M has infinite injective dimension.

The global dimension $\operatorname{gldim} A$ of the algebra A is def. as the supremum of the projective dimensions of all A-modules, that is, $\operatorname{gldim} A = \sup\{\operatorname{pd} M \mid M \in \operatorname{mod} A\}$

Remark: The global dimension of A can equivalently be def as the supremum of the injective dimensions of all A-modules.

Remark: 1. A module M is projective $\Leftrightarrow$ $\operatorname{pd} M = 0$
2. An algebra A is hereditary $\Leftrightarrow$ $\operatorname{gldim} A \leq 1$

EXAMPLE: The algebra given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \xrightarrow{\delta} 5$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$ is of global dimension 2. The fact that it is at least 2 can be seen by computing a minimal projective resolution for the simple module $S(1)$:

$$0 \longrightarrow P(3) \longrightarrow P(2) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$$

which shows that $\operatorname{pd} S(1) = 2$.

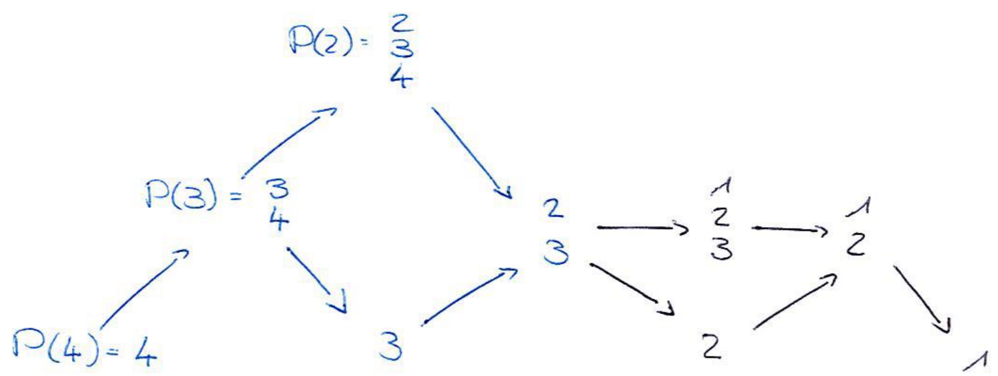
Chapter 5

In this chapter we give two examples of Auslander-Reiten quivers for bound quiver algebras. The main difference to the hereditary path algebras is that an arrow in the Auslander-Reiten quiver that ends at a projective module does not necessarily start at a projective module, but it does always start at an indecomposable direct summand of the radical of the projective.

Ex. I Let Q be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$ and $I = \langle \alpha\beta\gamma \rangle$. Here we will use the knitting algorithm to construct the Auslander-Reiten quiver.

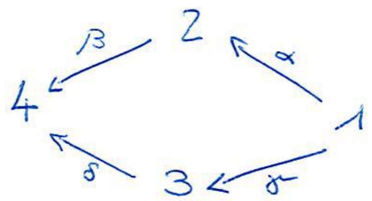
Step 1. The projective modules are: $P(1) = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$, $P(2) = \begin{smallmatrix}2\\3\\4\end{smallmatrix}$, $P(3) = \begin{smallmatrix}3\\4\end{smallmatrix}$, $P(4) = 4$.
And the radicals are $\mathrm{rad}\,P(1) = \begin{smallmatrix}2\\3\end{smallmatrix}$, $\mathrm{rad}\,P(2) = \begin{smallmatrix}3\\4\end{smallmatrix}$, $\mathrm{rad}\,P(3) = 4$, $\mathrm{rad}\,P(4) = 0$.

Step 2. With the use of the knitting algorithm, and the fact that we can draw an arrow from $P(4) \to P(3)$, since $P(4)$ is a direct summand of $P(3)$, we can draw it as follows:



Step 3. Now we can observe that $\begin{smallmatrix}2\\3\end{smallmatrix}$ is the $\mathrm{rad}\,P(1)$. So at this point we have to start a new τ-orbit, for $P(1)$, which is on the right of $\begin{smallmatrix}2\\3\end{smallmatrix}$ as in the Auslander-Reiten quivers of type D_n.
Then we can finish with the usual computations of the knitting algorithm.

Ex. II Let Q be the quiver



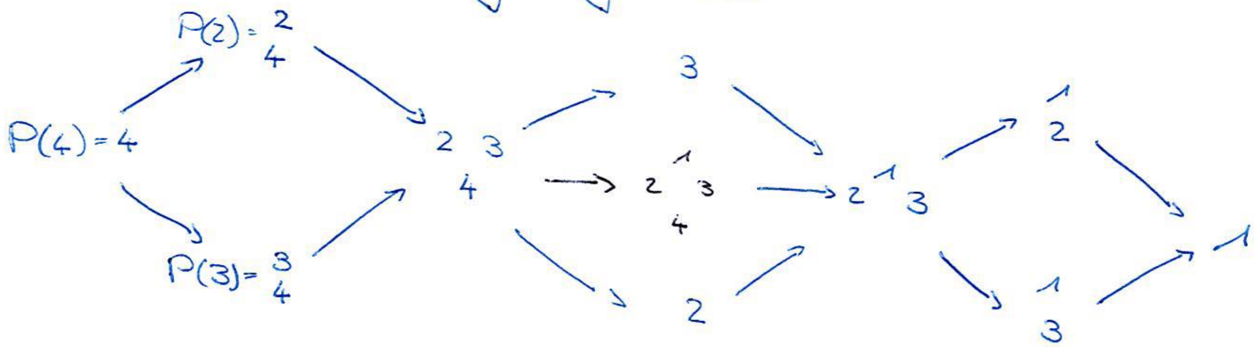
and $I = \langle \alpha\beta - \gamma\delta \rangle$

Step 1. $P(1) = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \end{smallmatrix}$, $P(2) = \begin{smallmatrix} 2 \\ 4 \end{smallmatrix}$, $P(3) = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}$, $P(4) = 4$

$\operatorname{rad} P(2) = \operatorname{rad} P(3) = P(4) = 4$

but $\operatorname{rad} P(1)$ is not projective.

Step 2. With the knitting algorithm:



But there is a problem, because $P(1)$ is missing. Because his radical is not projective. But we know that we can draw an arrow from $\begin{smallmatrix} 2 & & 3 \\ & 4 & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \end{smallmatrix}$ and so we can finish the Auslander-Reiten quiver.

PROBLEMS

1. Let Q be the quiver

$$\begin{tikzcd} & 2 \arrow[dl, "\beta"'] & \\ 4 & & 1 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & 3 \arrow[ul, "\delta"] & \end{tikzcd}$$

and def. two ideals $I_1 = \langle \alpha\beta + \gamma\delta \rangle$ and $I_2 = \langle \alpha\beta - \gamma\delta \rangle$. Show that

1. $I_1 \neq I_2$ unless the characteristic of k is 2.

   We clearly have that $\alpha\beta + \gamma\delta \neq \alpha\beta - \gamma\delta \iff \gamma\delta \neq -\gamma\delta$ if $\mathrm{char}(k) \neq 2$.

2. there exists an isomorphism of algebras $kQ/I_1 \longrightarrow kQ/I_2$

   In kQ/I_1 we have the relation $\alpha\beta + \gamma\delta = 0 \iff \alpha\beta = -\gamma\delta$. We note that $-\gamma\delta$ is an linear combination of the base $\gamma\delta$ and therefore a linear combination of $\alpha\beta$ since $\gamma\delta = -\alpha\beta$

   Basis of kQ/I_1 are: $B_1 = \{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \alpha\beta\}$

   In kQ/I_2 we have the relation $\alpha\beta - \gamma\delta = 0 \iff \alpha\beta = \gamma\delta$. So it follows that the bases of kQ/I_2 are:

   $B_2 := \{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \alpha\beta\}$

   $\Rightarrow B_1 = B_2 \Rightarrow$ There exists an isom. $kQ/I_1 \longrightarrow kQ/I_2$

2. Let Q be the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & 3 \arrow[r, "\delta"] & 4 \end{tikzcd}$$

and def. two ideals $I_1 = \langle \gamma\delta \rangle$ and $I_2 = \langle \gamma\delta - \alpha\beta\delta \rangle$

Show that $kQ/I_1 \cong kQ/I_2$

We first note that $I_1 \neq I_2$ and that the bases of kQ are:
$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \alpha\beta, \alpha\beta\delta, \gamma\delta\}$

In kQ/I_1 we have the relation $\gamma\delta = 0$, so it follows immediately that the bases of kQ/I_1 are
$B_1 = \{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \alpha\beta, \alpha\beta\delta\}$

In kQ/I_2 we have the relation $\gamma\delta - \alpha\beta\delta = 0 \iff \gamma\delta = \alpha\beta\delta$ so it follows that the bases kQ/I_2 are
$B_2 = \{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \alpha\beta, \alpha\beta\delta\}$

$\Rightarrow kQ/I_1$ and kQ/I_2 have the same bases $\Rightarrow kQ/I_1 \cong kQ/I_2$

3. Let Q be the quiver

$$\begin{tikzcd}
& 2 \arrow[dr, "\beta"] & \\
1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"] & & 4 \arrow[dl, "\delta"] \\
& 3 \arrow[ul, "\varepsilon"] &
\end{tikzcd}$$

and def. two ideals $I_1 = \langle \gamma\delta, \delta\varepsilon \rangle$ and $I_2 = \langle \gamma\delta - \alpha\beta\delta, \delta\varepsilon \rangle$. Show that $kQ/I_1 \cong kQ/I_2$

We first note that $I_1 \neq I_2$ and that the bases of kQ are
$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \varepsilon, \alpha\beta, \alpha\beta\delta, \alpha\beta\delta\varepsilon, \alpha\beta\delta\varepsilon\gamma, \beta\delta, \beta\delta\varepsilon, \beta\delta\varepsilon\alpha, \beta\delta\varepsilon\gamma, \delta\varepsilon, \delta\varepsilon\gamma, \delta\varepsilon\alpha, \delta\varepsilon\alpha\beta,$
$\varepsilon\gamma, \varepsilon\alpha, \varepsilon\gamma\delta, \varepsilon\alpha\beta, \varepsilon\alpha\beta\delta, \gamma\delta, \gamma\delta\varepsilon, \gamma\delta\varepsilon\alpha, \gamma\delta\varepsilon\alpha\beta\}$

In kQ/I_1 we have the relation $\gamma\delta = 0$ and $\delta\varepsilon = 0$. So it follows that the bases of kQ/I_1 are:
$B_1 = \{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \varepsilon, \alpha\beta, \alpha\beta\delta, \beta\delta, \varepsilon\gamma, \varepsilon\alpha, \varepsilon\alpha\beta, \varepsilon\alpha\beta\delta\}$

In kQ/I_2 we have the relation $\gamma\delta - \alpha\beta\delta = 0 \Leftrightarrow \gamma\delta = \alpha\beta\delta$ and $\delta\varepsilon = 0$. So it follows that the bases of kQ/I_2 are:

$B_2 = \{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \varepsilon, \alpha\beta, \alpha\beta\delta, \beta\delta, \varepsilon\gamma, \varepsilon\alpha, \varepsilon\alpha\beta\delta, \varepsilon\alpha\beta\}$

$\Rightarrow B_1 = B_2 \Rightarrow$ They have the same basis $\Rightarrow kQ/I_1 \cong kQ/I_2$

4. Give an example of a bound quiver algebra of global dimension 3

We take the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \xrightarrow{\delta} 5$$

and modify the relations to $\alpha\beta\gamma = 0$, $\gamma\delta = 0$. Then the minimal projective resolution for the simple module $S(1)$ becomes

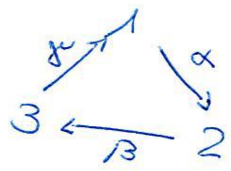
$$0 \longrightarrow P(5) \longrightarrow P(4) \longrightarrow P(2) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$$

which shows that $\mathrm{pd}\, S(1) = 3$. This algebra has global dimension 3.

Problem 5:

- Give an example of a bound quiver algebra of global dimension 4.

Let Q be the quiver



and $I = \langle \alpha\beta\gamma, \beta\gamma\alpha \rangle$

This has global dimension 4.

Problem 6:

- Give an example of a bound quiver algebra of infinite global dimension.

Let Q be the cyclic quiver with two vertices and two arrows α, β. Let $I = \langle \alpha\beta, \beta\alpha \rangle$.

This has infinite global dimension.