Gabriel's Theorem

Vivien Bammert & Mario Jung

29.04.2020

1 Representation type

1.1 Gabriel's Theorem: Finite Representation Type

In this talk we list the quivers of finite representation type. This classificication will only depend on the shape of the quiver and not on the particular orientation of the arrows. For this we need first the following definitions:

Definition 1. A quiver Q is of **finite representation type**, if the number of iso-classes of indecomposable representations of Q is finite.

Definition 2. The **underlying graph** of the quiver Q is the graph without the direction of the arrows. So we have the same vertices and for $i \to j$ we have $i - j$.

In the following picture are the so called Dynkin-Diagrams which are really important for our talk. We have that $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and $\mathbb{D}$ are infinite diagrams and the types $\mathbb{E}, \mathbb{F}$ and $\mathbb{G}$ are five exceptional diagrams. $\mathbb{A}, \mathbb{D}$ and $\mathbb{E}$ are called **simply laced** Dynkin diagrams, because they have no parallel edges.

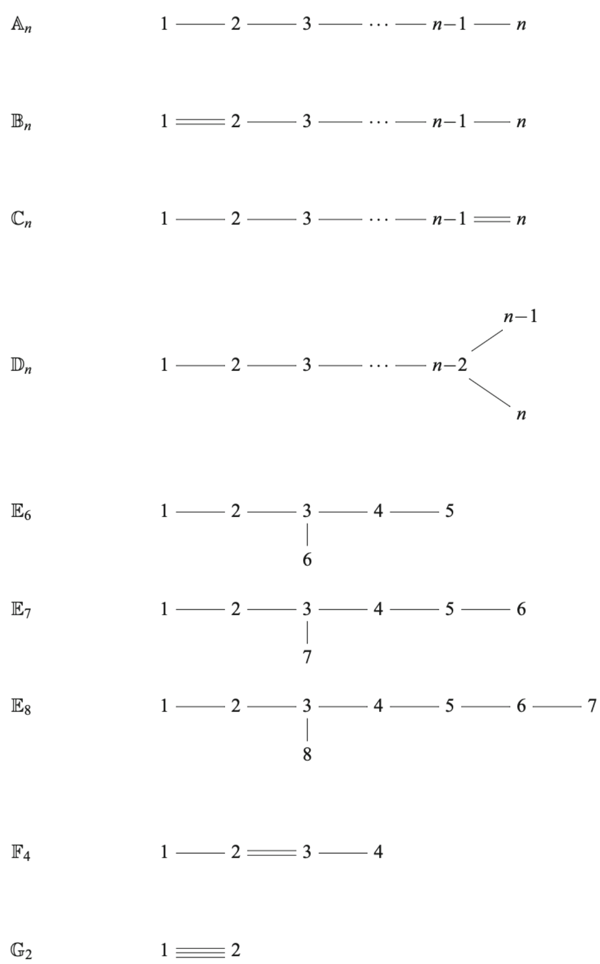


Figure 1: Dynkin diagrams

We are now able to state the main theorem of our talk:

Theorem 1 (Gabriel's Theorem)**.** *A connected quiver is of finite representation type if and only if its underlying graph is one of the Dynkin diagrams of type* $\mathbb{A}, \mathbb{D}$ *or* $\mathbb{E}$*.*

We will give a sketch of the proof at the end.
First we will concentrate on the Auslander-Reiten Quivers of Type $\mathbb{D}_n$.

2 Auslander–Reiten Quivers of Type $\mathbb{D}_n$

As in the last talk we use different techniques to compute the Auslander-Reiten quiver of Q of type $\mathbb{D}_n$. If a quiver Q is of type $\mathbb{D}_n$, then this means, that its underlying graph is the Dynkin diagram of type $\mathbb{D}_n$.

2.1 Knitting Algorithm

The Knitting-Algorithm for type $\mathbb{D}_n$ is almost the same as for type $\mathbb{A}_n$ with the difference that we need an additional fourth type of mesh:

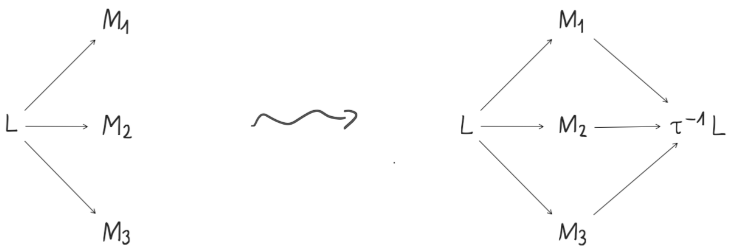


Figure 2: Knitting Algorithm

The dimension vectors $\boldsymbol{d} = (d_1, \ldots, d_n)$ which determine the isoclasses of indecomposable representations of quivers of type $\mathbb{D}_n$ are given by:

The entries d_i are either $0, 1$ or 2, and if we have $d_i = 2$, then

1. i is one of the vertices $2, 3, \ldots, n-2$,
2. for all vertices j with $i \leq j \leq n-2$ we have $d_j = 2$,
3. $d_{i-1} \geq 1$ and $d_{n-1} = d_n = 1$.

With the dimension vectors $\boldsymbol{d}$ we can construct the corresponding representation $M = (M_i, \varphi_\alpha)$, where $M_i = k^{d_i}$. We take $\varphi_\alpha = 1$ if $d_{s(\alpha)} = d_{t(\alpha)}$ and $\varphi_\alpha = 0$ if one of the $d_{s(\alpha)}, d_{t(\alpha)}$ is zero. For a vertex of dimension two we have exactly three arrows $\alpha_k, \beta_1, \beta_2$ which are between a vertex of dimension 1 and a vertex of dimension 2. With this three arrows we can look at the following three one-dimensional subspaces:

$$l_1 = \begin{cases} im(\varphi_{\alpha_k}) & \text{if } \alpha_k \text{ points to } k+1, \\ ker(\varphi_{\alpha_k}) & \text{otherwise.} \end{cases}$$

The one-dimensional subspace of M_{k+1}.

$$l_2 = \begin{cases} im(\varphi_{\beta_1}) & \text{if } \beta_1 \text{ points to } n-2, \\ ker(\varphi_{\beta_1}) & \text{otherwise.} \end{cases}$$

$$l_3 = \begin{cases} im(\varphi_{\beta_2}) & \text{if } \beta_2 \text{ points to } n-2, \\ ker(\varphi_{\beta_2}) & \text{otherwise.} \end{cases}$$

The one-dimensional subspaces of M_{n-2}.

Under the composition of the identity maps $\varphi_{\alpha_{n-3}} \cdots \varphi_{\alpha_{k+1}}$ l_1 is sent to a one-dimensional subspace $\tilde{l_1}$ of M_{n-2}.

Example 1. *Let Q be the quiver:*

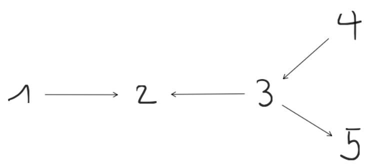


Figure 3: Quiver Q

Then

$$P(1) = \substack{1\\2},\ P(2) = 2,\ P(3) = \substack{3\\2\ \ 5},\ P(4) = \substack{4\\3\\2\ \ 5},\ P(5) = 5.$$

By using the Knitting-Algorithm we get the following Auslander-Reiten quiver:

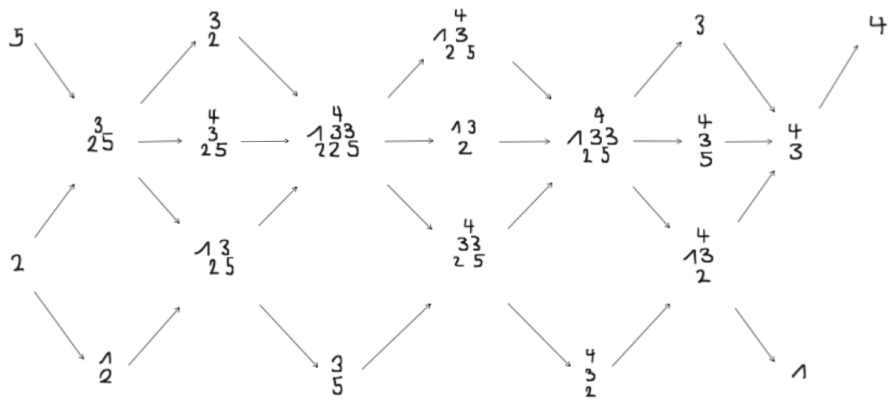


Figure 4: Knitting Algorithm for the Quiver Q

2.2 τ-Orbits

As we have seen for type $\mathbb{A}_n$ there are also several methods to compute the τ-orbits for type $\mathbb{D}_n$.

2.2.1 First Method: Auslander-Reiten Translation

The Auslander-Reiten Translation for type $\mathbb{D}_n$ is the same as for type $\mathbb{A}_n$. We do this shortly for *Example 1* with $M = \begin{smallmatrix}1\,3\\2\,5\end{smallmatrix}$ and we get $\tau^{-1}M = \begin{smallmatrix}4\\3\,3\\2\,5\end{smallmatrix}$.

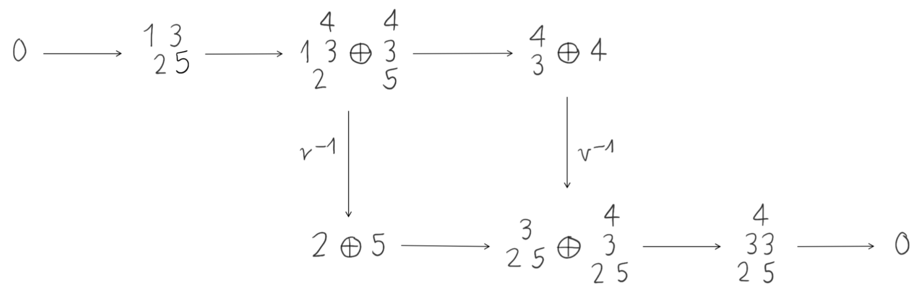


Figure 5: Auslander-Reiten Translation for M

2.2.2 Second Method: Coxeter Functor

Also the *Coxecter Functor Method* stays the same as for type $\mathbb{A}_n$.
The *cartan matrix* C and the *Coxecter matrix* $\Phi = -C^tC^{-1}$ stay also the same. In our *Example 1* we would get:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad (C^{-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\Phi = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \quad \Phi^{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

So we can compute the dimension vector of $\tau^{-1}M$ as follows: $\Phi^{-1}dim(M) = dim(\tau^{-1}M)$.
On the other hand: $\Phi dim(M) = \tau M$.

2.2.3 Arcs of punctured Polygons with n vertices

For a quiver Q of the type $\mathbb{D}_n$ we can also give a geometric construction similar to the construction of type $\mathbb{A}_n$.
But there are a few differences between the two constructions:

- Instead of triangulated polygons we use triangulated punctured polygons.

- Instead of diagonals in the polygon we need arcs.

For every punctured polygon with n boundary vertices we have exactly n^2 arcs:

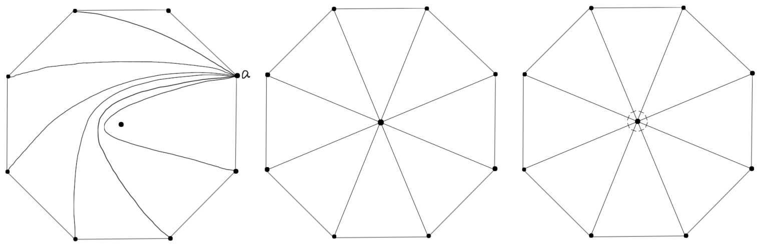


Figure 6: Arcs of a punctured polygon with eight boundary vertices

left: We have exactly $n-2$ arcs for every vertex a on the boundary. Since we have n vertices on the boundary we get the first $n^2 - 2n$ arcs.

middle: For every vertex on the boundary we have an arc to the puncture. So we get further n arcs.

right: For every vertex on the boundary we have another arc to the puncture. So we get the last n arcs.

So we have exactly n^2 arcs for the punctured polygon.
To distinguish between the n-arcs of the middle and right picture, we need a tag on the arcs. The ones with a tag are called *notched* and they without are called *plain*.

Remark 1. *If we have two boundary vertices $a \neq b$ s.t. they are neighbours, then we have exactly one arc that connects a and b. And if they aren't neighbours, we have exactly two arcs.*

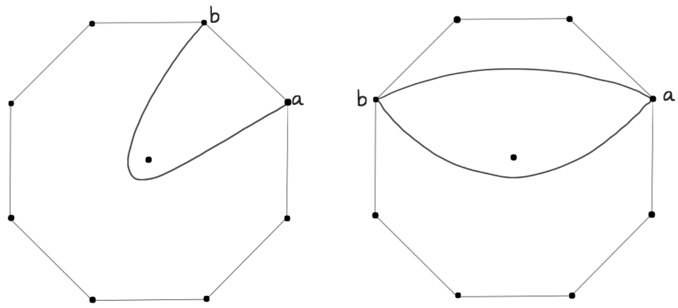


Figure 7: Arcs with specified endpoints

We now want to determine the number of crossings of two arcs γ, γ' which we denote by $e(\gamma, \gamma')$. It turns out, that it is not as straightforward to determine $e(\gamma, \gamma')$ in a punctured polygon as it is to say when two diagonals cross. For a rigorous definition of crossing numbers we would need the notion of homotopy, which will not be covered. So we will give a more intuitive definition:
If one or both arcs have both of their endpoints on the boundary, then it should be intuitively clear that $e(\gamma, \gamma')$ is either $0, 1, 2$. If γ and γ' are both incident to the puncture and a and a' are their endpoints on the boundary, we can define $e(\gamma, \gamma')$ as follows:

$$e(\gamma, \gamma') = \begin{cases} 0 & \text{if } \gamma \text{ and } \gamma' \text{ are both plain,} \\ 0 & \text{if } \gamma \text{ and } \gamma' \text{ are both notched,} \\ 0 & \text{if } a = a' \\ 1 & \text{if } \gamma, \gamma' \text{ have opposite tagging and } a \neq a'. \end{cases}$$

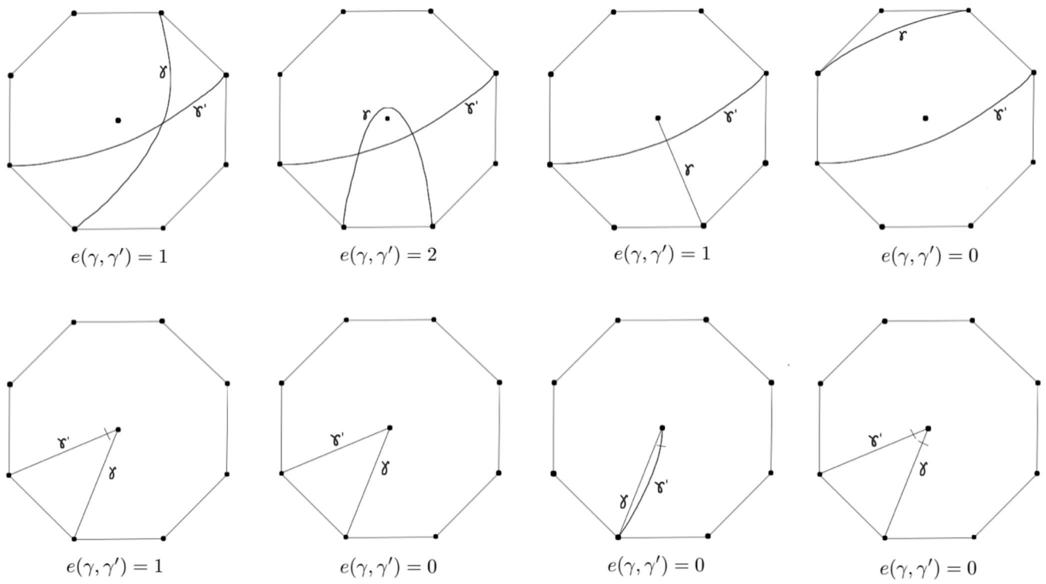


Figure 8: Crossing numbers

Definition 3. Let γ and γ' be two arcs. If $e(\gamma, \gamma') \geq 1$ then γ and γ' *cross.*

Definition 4. A *triangulation* T_Δ is a maximal set of non-crossing arcs.

Example 2. *We have here three triangulations of a 8-vertices polygon.*

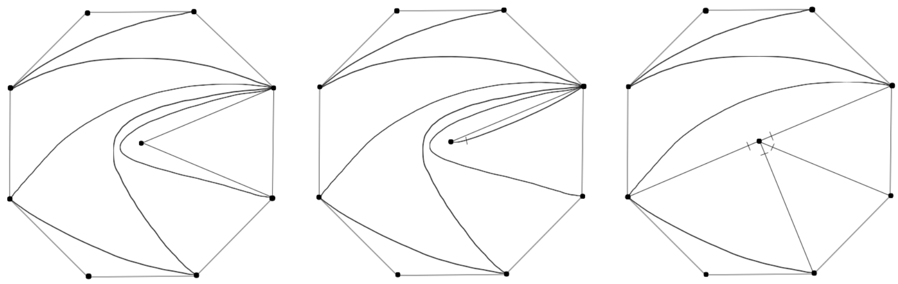


Figure 9: Examples of triangulations

Our next goal is to associate a Triangulation T_Q to Q. For this we need the following three steps:

1. We cut off a triangle $\triangle_0$ with an arc γ_1.

2. If $1 \leftarrow 2$ is in Q than we choose the unique γ_2 as follows:
   a) γ_2 forms the triangle $\triangle_1$ with γ_1 and a boundary segment.
   b) γ_1 is counterclockwise from γ_2 in $\triangle_1$.

   If $1 \rightarrow 2$ is in Q, then:
   a) γ_2 forms the triangle $\triangle_1$ with γ_1 and a boundary segment.
   b) γ_1 is clockwise from γ_2 in $\triangle_1$.

   Repeat step 2 until we have $n-2$ arcs.

3. We have two arcs left to determine: γ_{n-1}, γ_n. Depending on the orientation of the arrows, we choose them with the following four possibilities:

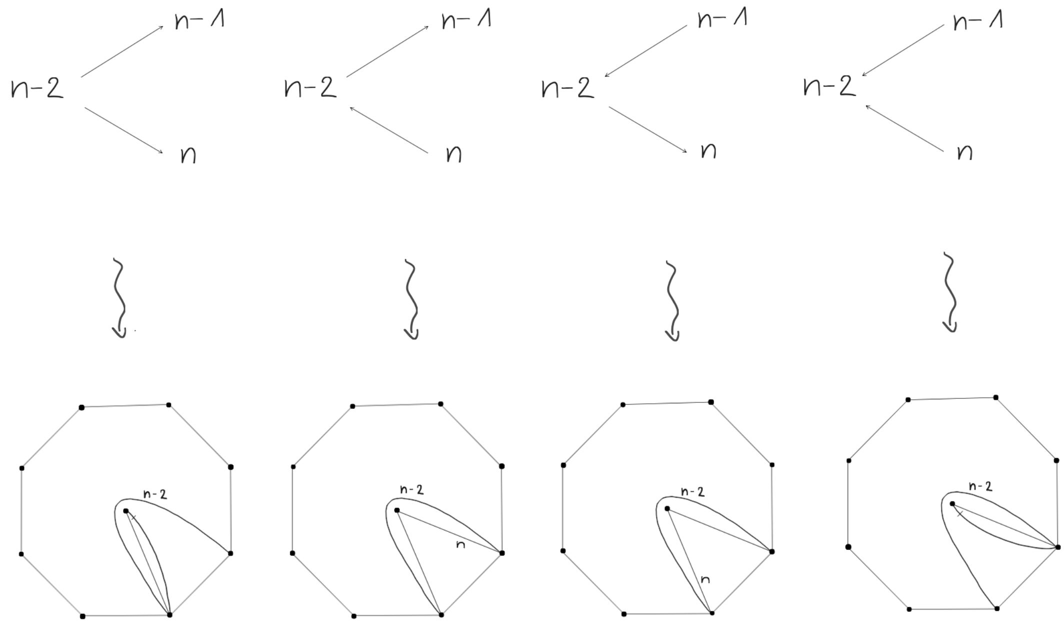
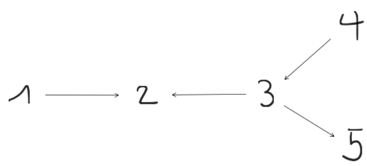
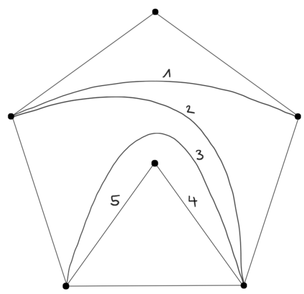


Figure 10: Four possibilities to construct the triangulation from Q

Example 3. *We now want to compute the triangulation from Q:*



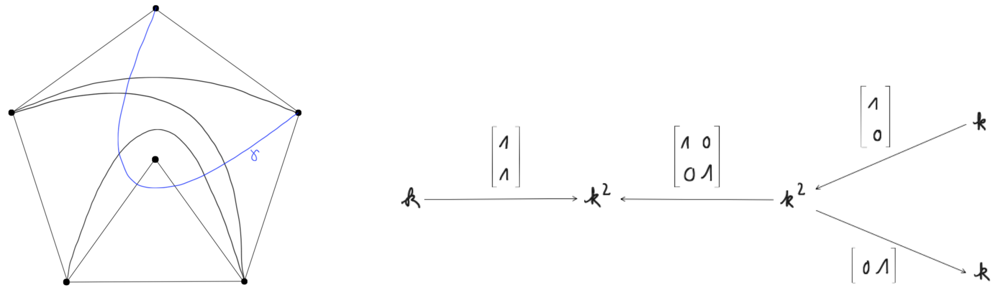
(a) Quiver Q



(b) Triangulation from the quiver Q

Figure 11: Construction of the Triangulation from the quiver Q

If we take an arc $\gamma \notin T_Q$, than we can associate the indecomposable representation $M_\gamma = (M_i, \varphi_\alpha)$ of Q as follows:
The dimension vector $\boldsymbol{d}$ is given by $d_i = e(\gamma, \gamma_i)$.



(a) Illustration of the crossings of γ and γ_i

(b) The isomorphic indecomposable representation of M_γ

Figure 12: Construction of the indecomposable representation M_γ

Remark 2. *The map $\gamma \mapsto M_\gamma$ is a bijection between the set of arcs that are not in T_Q and the set of isoclasses of indecomposable representations of Q.*

The Auslander-Reiten translation τ is given by an elementary clockwise rotation of the punctured polygon with simultaneous change of the tags at the puncture. So in our example the projective representation $P(i)$ is given by τ^{-1} of the arc γ_i, and the injective representation $I(i)$ is given by τ of the arc γ_i.

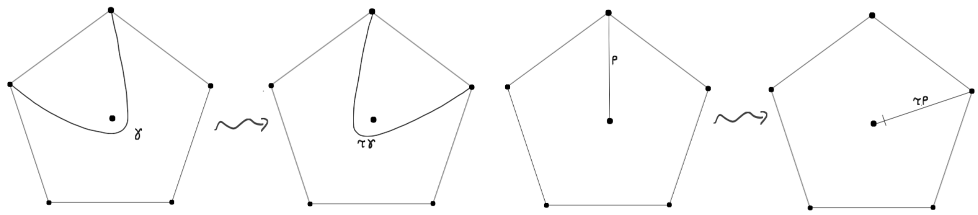


Figure 13: τ for arcs γ

Now we can construct the Auslander-Reiten quiver of our *Example 1* starting with the projectives and applying the elementary rotation to compute the τ-orbits until we reach the injective in each τ-orbit.

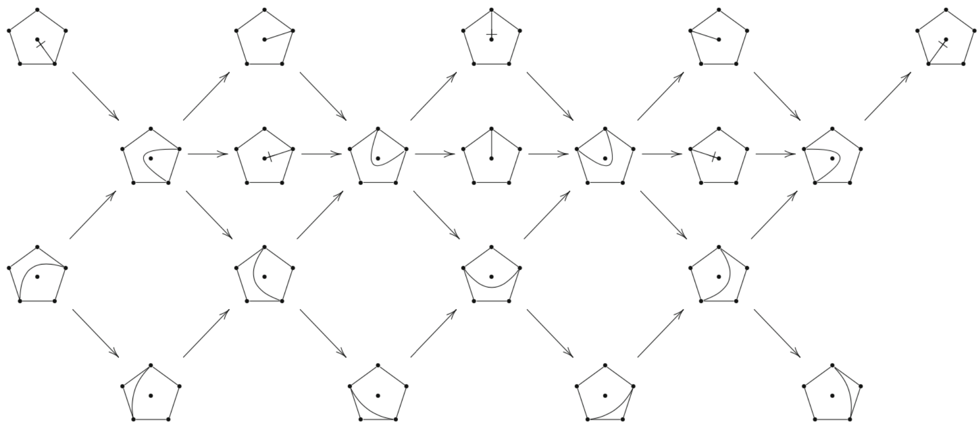


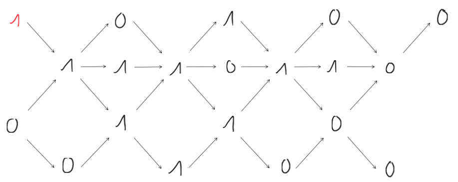
Figure 14: Auslander-Reiten quiver in terms of arcs in punctured polygons

2.2.4 Computing Hom Dimensions, Ext Dimensions, and Short Exact Sequences

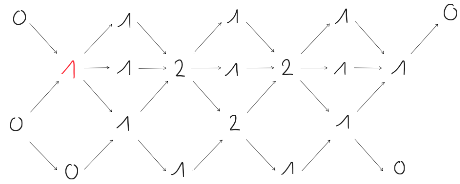
With the Auslander-Reiten quiver of type $\mathbb{D}_n$ we can compute the dimensions $Hom(M,N)$ and $Ext^1(M,N)$ as in type $\mathbb{A}$.

2.2.4.1 Dimension of Hom(M,N)

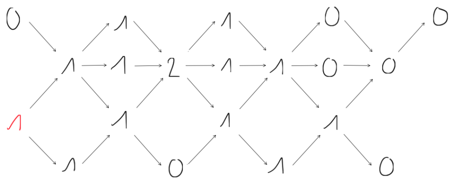
Let M, N be indecomposable representations of Q. As in type $\mathbb{A}$ the dimension of $Hom(M,N)$ is determined by the relative positions of the two representations in the Auslander-Reiten quiver. Instead of maximal slanted rectangles we have to use *hammocks*.
The definitions of a sectional path and the sets $\Sigma_{\rightarrow}(M)$ and $\Sigma_{\leftarrow}(M)$ stay the same. The hammock will be constructed by the following algorithm:
As in type $\mathbb{A}_n$ we start by labeling each vertex $\Sigma_{\rightarrow}(M)$ with the number 1. As a next step we consider the almost split sequence $0 \rightarrow M \rightarrow E \rightarrow \tau^{-1}M \rightarrow 0$. The summands of E lie in $\Sigma_{\rightarrow}(M)$ and $\tau^{-1}M$ does not. Then we can label the vertex $\tau^{-1}M$ by the sum of the labels of the indecomposable summands of E minus the label of M. So $\tau^{-1}M$ is $0, 1$ or 2 because of the $1, 2$ or 3 summands of E. We can construct these also recursively and if we get a label smaller then zero, we just take zero.



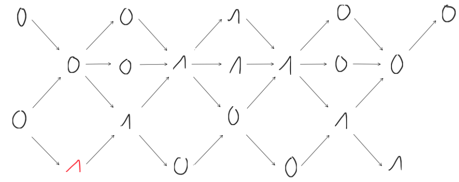
(a) Dimension of Hom(P(5), -)



(b) Dimension of Hom(P(3), -)



(c) Dimension of Hom(P(2), -)



(d) Dimension of Hom(P(1), -)

Figure 15: Dimensions of Hom(M, -) for M = P(1),P(2),P(3) and P(5)

2.2.4.2 $\boldsymbol{Ext^1(M,N)}$

To compute $Ext^1(M,N)$ we can also use the same formula as for type $\mathbb{A}_n$:

$$dim(Ext^1(M,N)) = dim(Hom(N,\tau M))$$

2.2.4.3 Short Exact Sequences

Finding short exact sequences that represent the elements of $Ext^1(M,N)$ is more difficult than for type $\mathbb{A}$. Because $dim(Ext^1(M,N))$ can be also two.
From the last talk we know that each element of $Ext^1(M,N)$ can be represented by a short exact sequence $0 \to N \to E \to M \to 0$ with E a representation of Q. But there can be more than only two choices for E. We illustrate that with an example:

Example 4. *In the figure 16 below there are four non-split short exact sequences which starts at N and ends at M:*

$$\begin{gathered} 0 \to N \to E_1 \oplus E_2 \oplus H_2 \to M \to 0 \\ 0 \to N \to F_1 \oplus F_2 \oplus H_2 \to M \to 0 \\ 0 \to N \to G_1 \oplus G_2 \to M \to 0 \\ 0 \to N \to H_1 \oplus H_2 \to M \to 0 \end{gathered}$$

Remark 3. *It is important to note that while there are four non-split short exact sequences, the dimension of $Ext^1(M,N)$ is only two. Thus any two of the above sequences span the vector space $Ext^1(M,N)$.*

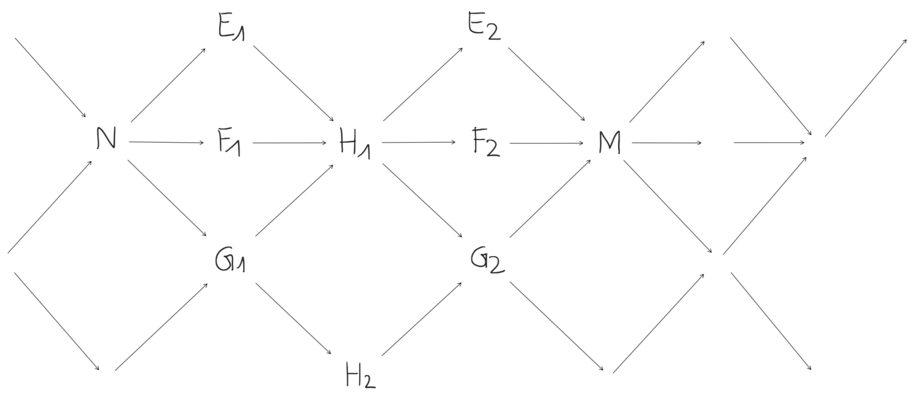


Figure 16: Computing short exact sequences

3 Gabriel's Theorem

To prove Gabriel's Theorem we have to define a quadratic form q associated to Q and we have to introduce its roots.

3.1 Quadratic forms

Definition 5. An *n-ary integral quadratic form* q is a homogeneous polynomial of degree 2 in n variables $x_1, x_2, \dots, x_n$ and with coefficients in $\mathbb{Z}$. So we can write q in the following way:

$$q(x_1, x_2, \dots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

with $a_{ij} \in \mathbb{Z}$.
We will often think of a quadratic form as a map:

$$q : \mathbb{Z}^n \to \mathbb{Z},\ \boldsymbol{x} = (x_1, \dots, x_n) \mapsto q(\boldsymbol{x}).$$

Remark 4. $q(r\boldsymbol{x}) = r^2 q(\boldsymbol{x})$, $\forall r \in \mathbb{Z}$

Definition 6. With the quadratic form q we can define its *symmetric bilinear form* $(\boldsymbol{x}, \boldsymbol{y})$ as follows:

$$(\boldsymbol{x}, \boldsymbol{y}) = q(\boldsymbol{x} + \boldsymbol{y}) - q(\boldsymbol{x}) - q(\boldsymbol{y}).$$

Remark 5. *So when we compute* $(\boldsymbol{x}, \boldsymbol{y})$*, we get*

$$(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i,j} a_{ij}(x_i y_j + x_j y_i)$$

Remark 6. *We can also recover $q(\boldsymbol{x})$ by:*

$$q(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x}, \boldsymbol{x})$$

Definition 7. Let Q be a quiver without oriented cycles, then we define its quadratic form by $q : \mathbb{Z}^n \to \mathbb{Z}$ with:

$$q(\boldsymbol{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}.$$

Remark 7. *q only depends on the underlying graph of Q.*

Example 5. *The quadratic form of the quiver*

$$1 \longrightarrow 2 \longleftarrow 3$$

is

$$q(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3$$

If we calculate q on the dimension vector $\boldsymbol{d}$ of a representation of Q, we can see, that the value of q only depends on the dimension vector and not on the particular representation itself.
This means that q is constant on E_d, where E_d denotes the space of all representations $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q with dimension vector $\boldsymbol{d}$.

Proposition 1. *For any representation M of dimension vector $\boldsymbol{d}$ we have $q(\boldsymbol{d}) = dim(Hom(M, M)) - dim(Ext^1(M, M))$.*

Next we will introduce some more notions about quadratic forms:

Definition 8. Let q be a quadratic form:

1. q is called *positive definite* if $q(\boldsymbol{x}) > 0$, for all $\boldsymbol{x} \neq 0$.
2. q is called *positive semi-definite* if $q(\boldsymbol{x}) \geq 0$, for all $\boldsymbol{x}$.

Lemma 1. *Assume that Q is connected. Let $\boldsymbol{d} = (d_i) \in \mathbb{Z}^n \backslash \{0\}$ be such that $(\boldsymbol{d}, \boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{Z}^n$. Then*

1. *q is positive semi-definite*
2. *$d_i \neq 0$ for all i*
3. *$q(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = \frac{a}{b}\boldsymbol{d}$, for some integers a, b.*

Definition 9. The *Euclidean diagrams* are defined in the picture below:

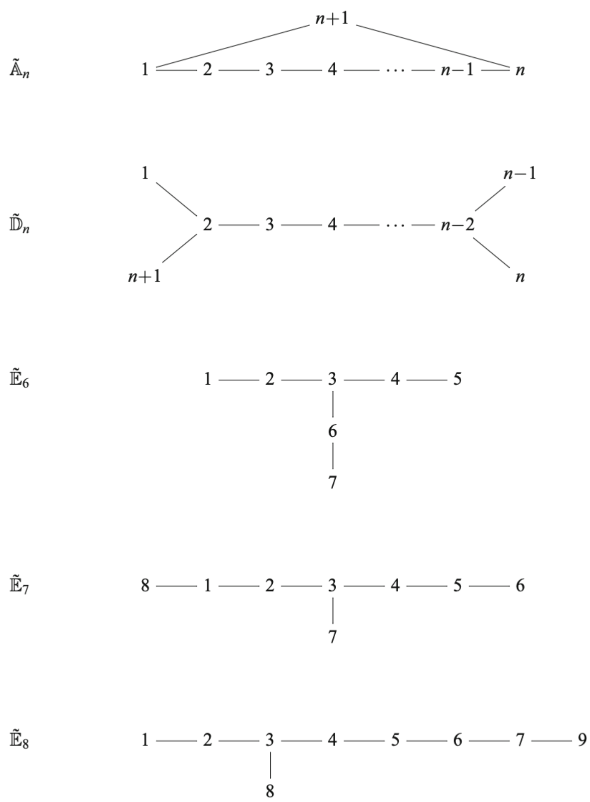


Figure 17: Euclidean diagrams

Now we are able to prove the main ingredient to understand Gabriel's Theorem:

Theorem 2. *Let Q be a connected quiver. Then*

1. *q is positive definite if and only if Q is of Dynkin type $\mathbb{A}, \mathbb{D}, \mathbb{E}$.*
2. *q is positive semi-definite if and only if Q is of Euclidean type $\tilde{\mathbb{A}}, \tilde{\mathbb{D}}, \tilde{\mathbb{E}}$, or of Dynkin type $\mathbb{A}, \mathbb{D}, \mathbb{E}$.*

Proof. ① Q **Euclidean** $\Rightarrow$ q **positive semi-definite.**
For every Euclidean Diagram we need to find a vector $\boldsymbol{\delta}$ s.t. $(\boldsymbol{\delta}, \boldsymbol{x}) = 0$ for all $\boldsymbol{x}$. Then with *Lemma 1* follows that q is positive semi-definite. The vectors $\boldsymbol{\delta}$ s.t. $(\boldsymbol{\delta}, \boldsymbol{x}) = 0$ for all $\boldsymbol{x}$ are shown in the following figure:

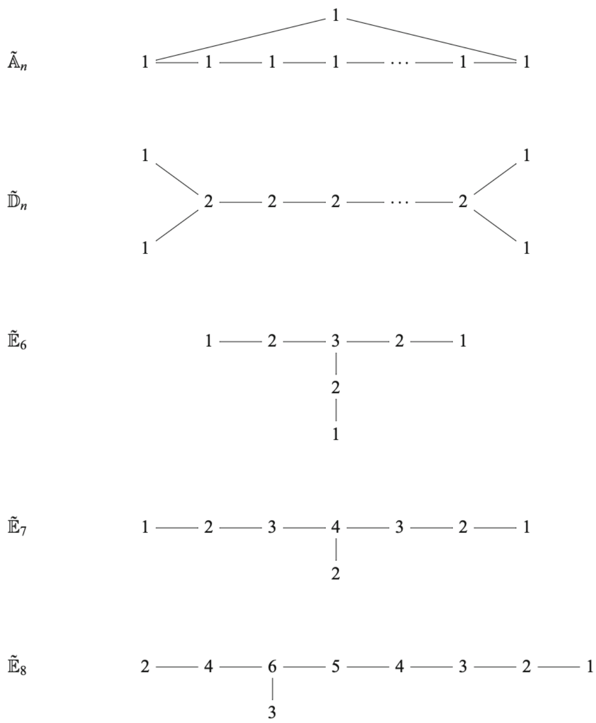


Figure 18: Vector $\boldsymbol{\delta}$ s.t. $(\boldsymbol{\delta}, \boldsymbol{x}) = 0$

② q **positive semi-definite** $\Rightarrow$ Q **Euclidean or** Q **Dynkin**
Assume Q is not Euclidean nor Dynkin. Then Q contains a proper subquiver Q' of Euclidean type. Let q' be the quadratic form of Q' and $\boldsymbol{\delta}$ be the dimension vector given in the figure above.

- If Q and Q' have the same set of vertices, then Q has more arrows than Q' $\Rightarrow 0 = q'(\boldsymbol{\delta}) > q(\boldsymbol{\delta})$. So we get a contradiction.
- If Q has more vertices than Q', then we can choose a vertex i_0 in Q s.t. $i_0 \to j_0$ with $j_0 \in Q'$. We define $\boldsymbol{x}$ by $x_i = 2\delta_i$ $\forall i \in Q'_0$, $x_{i_0} = 1$ and $x_j = 0$ for all other vertices j in Q.

$$\Rightarrow q(\boldsymbol{x}) - 1 + 2\delta_{j_0} = q'(2\boldsymbol{\delta})$$
$$\Leftrightarrow q(\boldsymbol{x}) = q'(2\boldsymbol{\delta}) + 1 - 2\delta_{j_0}$$

  Since $q'(2\boldsymbol{\delta}) = 4q'(\boldsymbol{\delta}) = 0 \Rightarrow q(\boldsymbol{x}) = 1 - 2\delta_{j_0} < 0$ we get a contradiction.

③ q **positive definite** $\Rightarrow$ Q **Dynkin**
For each Euclidean diagram we would have $q(\boldsymbol{\delta}) = 0$. $\Rightarrow q$ would not be positive

definite, so Q must be Dynkin.

④ Q **Dynkin** $\Rightarrow q$ **positive definite**
If we extend Q at one vertex labeled $n+1$, then we get an Euclidean quiver $\overline{Q}$ and $\overline{q}$ denotes its quadratic form.
Suppose: $\exists \boldsymbol{x} \in \mathbb{Z}^n \backslash \{0\}$ s.t. $q(x) \leq 0$.
Let $\overline{\boldsymbol{x}} \in \mathbb{Z}^{n+1}$, with $\overline{x_i} = x_i$ if $i \neq n+1$ and $\overline{x}_{n+1} = 0$
$\Rightarrow \overline{q}(\overline{\boldsymbol{x}}) = q(\boldsymbol{x}) \leq 0 \Rightarrow \overline{q}(\overline{\boldsymbol{x}}) = 0$ (since $\overline{q}$ is positive semi-definite.)
By *Lemma 1* it follows that $\overline{\boldsymbol{x}} = \frac{a}{b}\delta$ but this is not possible since $\overline{x}_{n+1} = 0$.
$\Rightarrow q$ is positive definite. $\square$

3.2 Roots

For a positive semi-definite quadratic form, there are two kind of roots:

Definition 10. Let $\boldsymbol{x} \in \mathbb{Z}^n \backslash \{0\}$.

- If $q(\boldsymbol{x}) = 1$, then $\boldsymbol{x}$ is called *real root.*
- If $q(\boldsymbol{x}) = 0$, then $\boldsymbol{x}$ is called *imaginary root.*

Remark 8. *Every root α is of the form $\alpha = \sum_i a_i \boldsymbol{e_i}$ with $a_i \in \mathbb{Z}$ and $\boldsymbol{e_i}$ the standard basis vector in $\mathbb{Z}^n$.*

Definition 11. Let $\alpha = \sum_i a_i \boldsymbol{e_i}$ be a root.

- α is called *positive* if all $a_i \geq 0$.
- α is called *negative* if all $a_i \leq 0$.

Let Φ be the set of all roots. Φ_+ the set of all positive roots and Φ_- the set of all negative roots.

Remark 9. *If q is positive semi-definite, then each root is either positive or negative and $\Phi = \Phi_- \sqcup \Phi_+$, and $\Phi_- = -\Phi_+$. (Would also hold for q not positive semi-definite)*

Corollary 1. *If Q is of Dynkin type, then there are finitely many roots and each root is a real root.*

In order to understand the proof of Gabriel's Theorem we need a few more notions.

Remark 10. *We need the notion of an orbit:*

$$\mathcal{O}_M = \{M' \in rep(Q) \mid M' \cong M\}.$$

Lemma 2. *Let $\boldsymbol{d} \in \mathbb{Z}^n$. Then there is at most one orbit $\mathcal{O}$ of codimension zero in E_d.*

Lemma 3. *If*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is a non-split short exact sequence of representations, then

$$dim(\mathcal{O}_{L\oplus N}) < dim(\mathcal{O}_M)$$

Proposition 2. *Let Q be a connected quiver and let M be a representation of Q of dimension vector $\boldsymbol{d}$. Then*

$$codim(\mathcal{O}_M) = dim(End(M)) - q(d) = dim(Ext^1(M, M))$$

Remark 11. $codim(\mathcal{O}_M) = dim(E_d) - dim(\mathcal{O}_M)$

Corollary 2. *If $q(\boldsymbol{d}) \leq 0$ then there are infinitely many isoclasses of representations of Q of dimension vector $\boldsymbol{d}$.*

Proof. Let $\boldsymbol{d}$ s.t. $q(\boldsymbol{d}) \leq 0$ and let M be a representation of Q s.t. $dim(M) = \boldsymbol{d}$. Then by *Proposition 2* it follows, that:

$codim(\mathcal{O}_M) \geq dim(End(M)) \geq 1$ and by *Remark 11* follows that $dim(E_d) > dim(\mathcal{O}_M)$

This shows that we have infinitely many isoclasses of representations of Q. □

Now we are ready to give a sketch of the proof of Gabriel's Theorem.

Theorem 3 (Gabriel's Theorem)**.** *Let Q be connected quiver. Then*

1. *Q is of finite representation type if and only if Q is of Dynkin type $\mathbb{A}, \mathbb{D}$ or $\mathbb{E}$.*
2. *If Q is of Dynkin type $\mathbb{A}, \mathbb{D}$ or $\mathbb{E}$, then the dimension vector induces a bijection ψ from isoclasses of indecomposable representations of Q to the set of positive roots:*

$$\psi : ind(Q) \longrightarrow \Phi_+ \qquad\qquad \psi(M) = dim(M)$$

Proof. We first sketch part (2), since we will need it for part (1):
(2) Note that since Q is of Dynkin type $\mathbb{A}, \mathbb{D}, \mathbb{E}$ we know from *Theorem 2* that q is positive-definite.
First we show that ψ is well-defined: We take M as an indecomposable representation of Q, and show that $q(dim(M)) = 1$. From *Proposition 1* we know that:

$$q(\boldsymbol{d}) = dim(Hom(M, M)) - dim(Ext^1(M, M))$$

This implies, that we only need to show that $End(M) \cong k$ and $Ext^1(M, M) = 0$ to get that $q(\boldsymbol{d}) = 1$.

To show that $End(M) \cong k$ we do an induction on the dimension of M:
For the simple representation it follows that $End(M) \cong k$.
To show that it holds for $dim(M) > 1$, we assume $End(M) \ncong k$ and $End(L) \cong k$ for all proper subrepresentations L of M.
With the knowledge of talk 9, that every endomorphism of M can be written as $\lambda 1_M + g$ for some $\lambda \in k$ and some nilpotent endomorphism g, we can define the following map i:

$$\begin{tikzcd} \mathrm{im}\, g \arrow[r, "incl."] \arrow[rr, bend right, "i"'] & \ker g \arrow[r, "\pi"] & L \end{tikzcd}$$

with the projection map π.
By taking g s.t $im(g)$ has minimal dimension, we can show over

$$\begin{tikzcd} M \arrow[r, "g"] & \mathrm{im}\, g \arrow[r, "i"] & L \arrow[r, "incl."] & M \end{tikzcd}$$

that i is injective and define the short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{im}\, g \arrow[r, "i"] & L \arrow[r] & \mathrm{coker}\, i \arrow[r] & 0 \end{tikzcd}$$

By applying the functor $Hom(-, L)$ we get the surjective morphism

$$\begin{tikzcd} \mathrm{Ext}^1(L, L) \arrow[r] & \mathrm{Ext}^1(\mathrm{im}\, g, L) \arrow[r] & 0 \end{tikzcd}$$

with which we can conclude by *Proposition 1* that $Ext^1(im(g), L) = 0$.
By considering the following diagram we get from $Ext^1(im(g), L) = 0$, that the bottom row splits.

$$\begin{tikzcd} 0 \arrow[r] & \ker g \arrow[r, "u"] \arrow[d, "\pi"] & M \arrow[r, "g"] \arrow[d, "j_2"] & \mathrm{im}\, g \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "j_1"] & X \arrow[r] & \mathrm{im}\, g \arrow[r] & 0 \end{tikzcd}$$

By using this fact, we can conclude that $L = M$ or $L = 0$.

But $im(g) \cap L \neq 0 \Rightarrow L \neq 0$ and $L \subset ker(g)$ with g non-zero $\Rightarrow L \neq M$. So we get $dim(End(M)) = 1$. Again with *Proposition 1* we finally get that $dim(M)$ is a positive root and ψ is well defined.
Now it only remains to show that ψ is bijective.

- ψ **is injective** because if we take M, M' two indecomposable representations s.t. $dim(M) = dim(M')$, $Ext^1(M, M) = 0$ from above, then by *Proposition 2* $codim(\mathcal{O}_M) = codim(\mathcal{O}_{M'}) = 0$ and so $M \cong M'$.

- To show that ψ is **surjective** we take a representation M s.t. $dim(M) = \boldsymbol{d}$ and the orbit $\mathcal{O}_M$ of maximal dimension in E_d and show that M is indecomposable. We do this by contradiction:
  Let $M = M_1 \oplus M_2$ then if $Ext^1(M_1, M_2) = Ext^1(M_2, M_1) = 0$ it follows from *Proposition 1* that

  $$1 = q(\boldsymbol{d}) = dim(Hom(M_1 \oplus M_2, M_1 \oplus M_2)) \geq 2 \Rightarrow \text{ Contradiction}$$

  So M is indecomposable, $\psi(M) = \boldsymbol{d} \Rightarrow \psi$ surjective.
  If $Ext^1(M_1, M_2) \neq 0$ then there would be a non-split short exact sequence

  $$0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

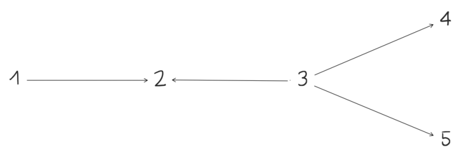
  Then by *Lemma 3* follows that $dim(\mathcal{O}_M) < dim(\mathcal{O}_E)$. But this is a contradiction due to the maximality of $\mathcal{O}_M$. So $Ext^1(M_1, M_2) = Ext^1(M_2, M_1) = 0$.

$\Rightarrow$ part (2) is proved.
Now we can finally prove the first part:
"$\Rightarrow$" Assume Q is not of Dynkin type $\mathbb{A}, \mathbb{D}, \mathbb{E}$, then by *Theorem 2 part 1* it follows that $\exists \boldsymbol{d} \neq 0$ s.t. $q(\boldsymbol{d}) \leq 0$. By *Corollary 2* follows then, that there are infinitely many isoclasses of representations of Q of dimension vector $\boldsymbol{d} \Rightarrow Q$ is not of finite representation type.
"$\Leftarrow$" If Q is of Dynkin type $\mathbb{A}, \mathbb{D}, \mathbb{E}$ then we know by part (2) that we have a bijection between $ind(Q)$ and Φ_+.
By *Corollary 1* it follows that there are finitely many roots of q and so Φ_+ is also finite.
$\Rightarrow ind(Q)$ is also finite.
$\Rightarrow Q$ is of finite representation type. $\square$

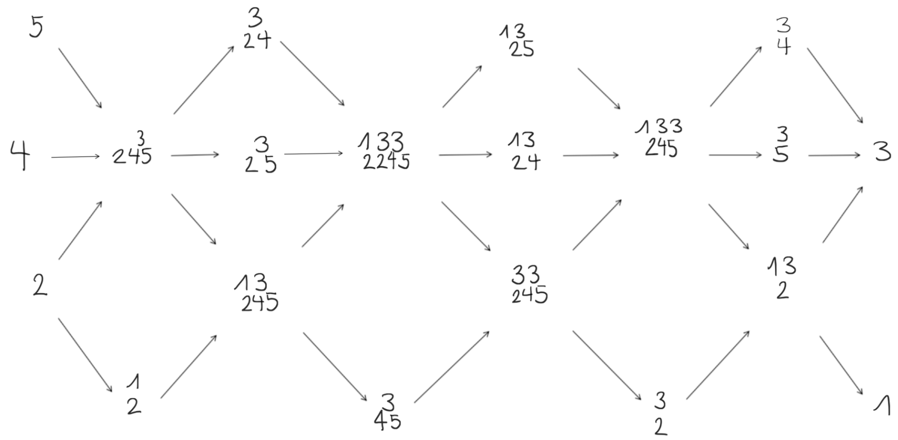
4 Problems

4.1 Problem 1

We want to compute the Auslander-Reiten quiver of the following quiver Q:

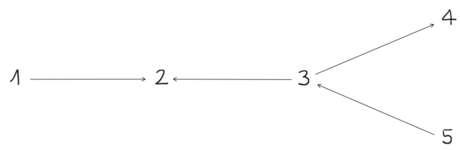


So we get with $P(1) = {}^{1}_{2}, P(2) = 2, P(3) = {}^{\ 3}_{245}, P(4) = 4$ and $P(5) = 5$:

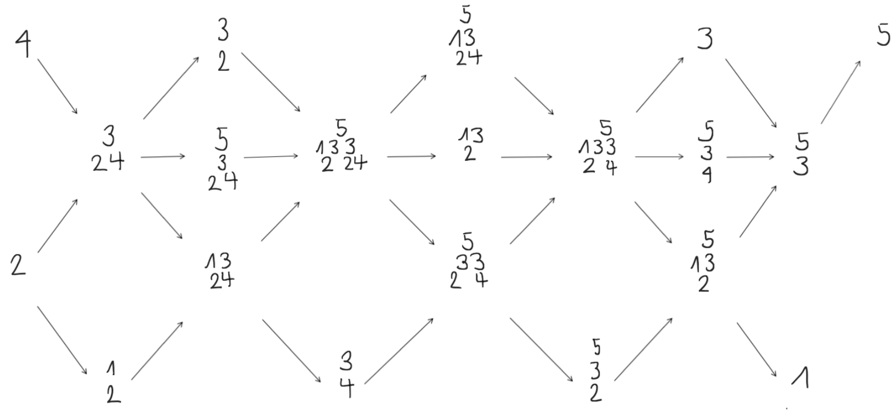


4.2 Problem 2

We want to compute the Auslander-Reiten quiver of the following quiver Q:



So we get with $P(1) = \begin{smallmatrix}1\\2\end{smallmatrix}, P(2) = 2, P(3) = \begin{smallmatrix}&3&\\2&&4\end{smallmatrix}, P(4) = 4$ and $P(5) = \begin{smallmatrix}&5&\\&3&\\2&&4\end{smallmatrix}$:



5 Literatur

All the informations are from the book:
Quiver Representations from Ralf Schiffler.