

# Math564: Representation theory of $\mathfrak{sl}_2$

## Talk 1: The finite-dimensional case I - the simples

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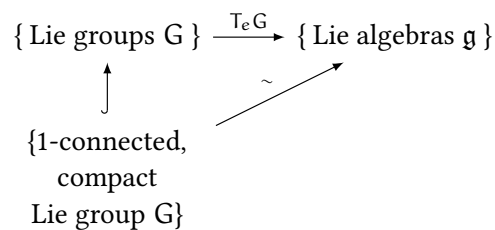
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### Abstract

The aim of these notes is to make the life of the other participants easier, if my talk gets too chaotic at some point.

## 1 General motivation

Studying the theory of Lie algebras allows us to understand the structure of more complicated mathematical objects, called Lie groups. We can illustrate the correspondence between Lie groups and Lie algebras with the following diagram:



The main object of study of the seminar will be one of the smallest but important Lie algebras  $\mathfrak{sl}_2$ , corresponding to the Lie group  $SL_2$ . We're interested in its representation theory for many reasons, e.g.:

- $\mathfrak{sl}_2(\mathbb{C})$  is isomorphic to the complexification of the real Lie algebras  $\mathfrak{so}_3$  and  $\mathfrak{su}_2$ , i.e.  $\mathfrak{so}_3 \otimes \mathbb{C}$ , respectively  $\mathfrak{su}_2 \otimes \mathbb{C}$ ; **(not a subject of the seminar)**
- $\mathfrak{sl}_2(\mathbb{C})$  viewed as a six-dimensional real Lie algebra, is isomorphic to the Lie algebra of the Lorentz group, i.e. the group of linear transformations of the Minkowski space-time; **(not a subject of the seminar)**
- the representation theory of  $\mathfrak{sl}_2(\mathbb{C})$  extends to other setups, namely the representation theory of semisimple Lie algebras  $\mathfrak{g}$  can be studied using the same tricks as for  $\mathfrak{sl}_2(\mathbb{C})$  with minimal additional knowledge since everything in  $\mathfrak{sl}_2(\mathbb{C})$  is computable and known! **(subject of the seminar)**

## 2 Basic definitions

**Assumption 2.1.** Throughout these notes we will always work over the field of complex numbers  $\mathbb{C}$ , unless it is said that we work over another field.

**Definition 2.2.** The Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$  consists of the vector space

$$\mathfrak{sl}_2(\mathbb{C}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C} \text{ and } a + d = 0 \right\}$$

of all complex  $2 \times 2$  matrices with zero trace and binary bilinear operation, called Lie bracket, and defined as  $[X, Y] := XY - YX$  for  $X, Y$  arbitrary matrices.

**Remark 2.3.** Taking the Lie bracket of two matrices is a well-defined operation. Indeed, take two arbitrary  $n \times n$  matrices  $A$  and  $B$  and apply the cyclicity of the trace to their Lie bracket:

$$\text{tr}([A, B]) = \text{tr}(AB) - \text{tr}(BA) = 0.$$

The lemma below characterizes every general Lie algebra, in fact it is a part of the definition of a Lie algebra.

**Lemma 2.4. (a)** For any  $X \in \mathfrak{g}$  we have  $[X, X] = 0$ .

**(b)** For any  $X, Y, Z \in \mathfrak{g}$  the Jacobi identity holds true, i.e.  $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ .

*Proof.* **(a)** Obviously,  $[X, X] = XX - XX = 0$ .

**(b)**  $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = [X, YZ - ZY] + [Y, ZX - XZ] + [Z, XY - YX] = X(YZ - ZY) - (YZ - ZY)X + Y(ZX - XZ) - (ZX - XZ)Y + Z(XY - YX) - (XY - YX)Z = XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY + ZXY - ZYX - XYZ + YXZ = 0$ .  $\square$

**Remark 2.5.**  $[X, X] = 0$  is equivalent to the property, called antisymmetry, defined as  $[X, Y] = -[Y, X]$  for all  $X, Y \in \mathfrak{g}$ . To see that, recall that  $[-, -]$  is bilinear and set  $0 = [X + Y, X + Y] = [X, X] + [X, Y] + [Y, X] + [Y, Y]$ . This is true for any field  $K$  with  $\text{char}(K) \neq 2$ .

The way we presented the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$  is not convenient for our purpose to study the representation theory of this algebra (infinitely many elements), that's why it is better to look at  $\mathfrak{sl}_2(\mathbb{C})$  in terms of generators and relations.

**Definition 2.6.** The Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$  is 3-dimensional and has a natural basis given by the matrices:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

subject of the relations:

$$[e, f] = ef - fe = h, \quad [h, e] = he - eh = 2e, \quad [h, f] = hf - fh = -2f. \quad (2.1)$$

**Assumption 2.7.** From now on, we will use the following notational convention:  $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{g}$ .

**Definition 2.8.** A module over  $\mathfrak{g}$  (or simply a  $\mathfrak{g}$ -module) is a vector space  $V$  together with three fixed linear operators  $E = E_V, F = F_V, H = H_V$  on  $V$ , which satisfy the equations from the relations, namely

$$EF - FE = H, \quad HE - EH = 2E, \quad HF - FH = -2F. \quad (2.2)$$

**Remark 2.9.** We can rewrite the last two relations as:

$$HE = 2E + EH = E(H + 2), \quad HF = FH - 2F = F(H - 2).$$

**Example 2.10.** Let  $V = \mathbb{C}$  and  $E = F = H = 0$ . This is the notion of the trivial  $\mathfrak{g}$ -module.

**Example 2.11.** Let  $V = \mathbb{C}^2$ . We identify the set of all linear operators on  $V$  with the set of all  $2 \times 2$  matrices. In this case the linear operators  $E, F$  and  $H$  coincide with the matrices  $e, f, h$ . The equations 2.1 and 2.2 are satisfied. This is how the natural (or standard)  $\mathfrak{g}$ -module is defined.

**Example 2.12.** Take  $V = \mathfrak{g}$ . We define the linear operators as follows:

$$E_{\text{adj}} \text{ is } [e, -], \quad F_{\text{adj}} \text{ is } [f, -], \quad H_{\text{adj}} \text{ is } [h, -].$$

This defines the adjoint  $\mathfrak{g}$ -module. To see this, check if the relations in 2.2 for  $\mathfrak{g}$  are satisfied for the corresponding matrices to the linear operators. The matrices are given by:

$$E_{\text{adj}} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad F_{\text{adj}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}, \quad H_{\text{adj}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**Lemma 2.13.** For any  $X \in \mathfrak{g}$  we have the following identities:

- (a)  $[e, [f, X]] - [f, [e, X]] = [h, X]$ ,
- (b)  $[h, [e, X]] - [e, [h, X]] = [2e, X]$ ,
- (c)  $[h, [f, X]] - [f, [h, X]] = [-2f, X]$ .

*Proof.* We prove only (a), since (b) and (c) follow the same idea.

We rewrite the equation from (a) as  $[e, [f, X]] - [f, [X, e]] - [h, X] = 0$ . Using that  $[e, f] = h$  and the antisymmetry of the Lie bracket, we obtain  $[e, [f, X]] + [f, [X, e]] + [X, [e, f]] = 0$ , which corresponds to the Jacobi identity from Lemma 2.4.

□

**Definition 2.14.** Let  $V, W$  be  $\mathfrak{g}$ -modules, then a  $\mathfrak{g}$ -homomorphism (also known as intertwiner) is a linear map  $\Phi : V \rightarrow W$ , which makes the diagram commute for all  $X \in \{E, F, H\}$ :

$$\begin{array}{ccc} V & \xrightarrow{X_V} & V \\ \downarrow \Phi & & \downarrow \Phi \\ W & \xrightarrow{X_W} & W. \end{array}$$

In other words  $X_W \circ \Phi = \Phi \circ X_V$ , i.e.  $\Phi$  intertwines the actions of  $e$ ,  $f$  and  $h$  on  $V$ , such that we get:

$$\Phi E_V = E_W \Phi, \Phi F_V = F_W \Phi, \Phi H_V = H_W \Phi.$$

**Example 2.15.** We call the morphism  $\Phi : V \rightarrow W$  defined as  $v \mapsto 0_W$ , the zero morphism. It clearly satisfies the condition of Definition 2.14, so it is a  $\mathfrak{g}$ -morphism.

**Example 2.16.** For any  $\mathfrak{g}$ -module  $V$ , the identity map  $\text{id}_V$  on  $V$  is an intertwiner. This is known as the identity morphism.

**Remark 2.17.** Let us denote the space of all  $\mathfrak{g}$ -homomorphisms as  $\text{Hom}_{\mathfrak{g}}(V, W)$ . It has the structure of a vector space. Let  $f, g \in \text{Hom}_{\mathfrak{g}}(V, W)$ , define addition as  $f + g : V \rightarrow W$ ,  $v \mapsto f(v) + g(v)$ . For scalar multiplication we have  $\alpha f : V \rightarrow W$ ,  $v \mapsto \alpha f(v) \in W$ . It follows that  $\text{Hom}_{\mathfrak{g}}(V, W) \neq 0$  and  $\dim(\text{Hom}_{\mathfrak{g}}(V, W)) = \dim(V)\dim(W)$ .

**Remark 2.18.** As seen in the (linear) algebra class we have the following notions for intertwiners:

- monomorphism is an injective  $\mathfrak{g}$ -homomorphism;
- epimorphism is a surjective  $\mathfrak{g}$ -homomorphism;
- isomorphism is a bijective  $\mathfrak{g}$ -homomorphism.

For us it makes sense to study  $\mathfrak{g}$ -modules up to isomorphism.

**Definition 2.19.** Let  $V$  be a  $\mathfrak{g}$ -module. A subspace  $W \subset V$  is called a subspace (or a  $\mathfrak{g}$ -submodule) of  $V$ , if it is invariant with respect to the action of  $E_V, F_V$  and  $H_V$ , that means:

$$E_V W \subset W, F_V W \subset W, H_V W \subset W.$$

**Remark 2.20.** The module  $V$  has always two submodules, namely the zero module  $\{0\}$  and  $V$  itself.

**Definition 2.21.** Any submodule that is not the zero submodule and not  $V$  itself is called a proper submodule. A module  $V$  that has no proper submodules is called simple.

**Example 2.22.** Any module of dimension one is simple.

**Example 2.23.** The trivial, natural and the adjoint modules are simple.

**Example 2.24.** Let  $V = \mathbb{C}^2$  and  $E = F = H = 0$ .  $V$  has the structure of a  $\mathfrak{g}$ -module which is not simple, e.g. there is a proper one dimensional submodule spanned by the vector  $e_1 = (1, 0)$ , we have  $E(e_1) = F(e_1) = H(e_1) = 0$ .

**Example 2.25.** Let  $V$  be a  $\mathfrak{g}$ -module and  $W$  be a submodule of  $V$ . The quotient space  $V/W$  has the natural structure of a  $\mathfrak{g}$ -module given by  $E(v+W) = E(v) + W$ ,  $F(v+W) = F(v) + W$ ,  $H(v+W) = H(v) + W$ .

As next we will see a lemma, which we will apply in the proof of the Schur's lemma.

**Lemma 2.26.** Let  $V$  and  $W$  be two  $\mathfrak{g}$ -modules and  $\Phi$  an intertwiner. Then we have:

(a)  $\text{Ker}(\Phi)$  of  $\Phi$  is a submodule of  $V$ .

(b) The image  $\text{Im}(\Phi)$  of  $\Phi$  is a submodule of  $W$ .

*Proof.* (a) Let  $v \in \text{Ker}(\Phi)$  and  $X \in \{E, F, H\}$ . We want to show that the space  $\text{Ker}(\Phi)$  is invariant under the action of  $X$ . Indeed:

$$\Phi(X_V(v)) = \Phi \circ X_V(v) = X_W \circ \Phi(v) = 0 \Rightarrow X_V(v) \in \text{Ker}(\Phi).$$

(b) Let  $w \in \text{Im}(\Phi)$ , then there exists some  $v \in V$  such that  $w = \Phi(v)$ . Apply the definition of an intertwiner:

$$\Phi \circ X_V(v) = X_W \circ \Phi(v) = X_W(w) \in \text{Im}(\Phi),$$

which proves the claim. □

### 3 Classification of simple finite-dimensional modules

In this section we will classify all simple finite-dimensional  $\mathfrak{g}$ -modules. We will see later that these modules form only a small family of the simple  $\mathfrak{g}$ -modules.

Throughout this section,  $V$  is always considered to be a non-zero finite-dimensional  $\mathfrak{g}$ -module. For  $\lambda \in \mathbb{C}$  we set:

$V(\lambda) = \{v \in V : (H - \lambda)^k v = 0 \text{ for some } k \in \mathbb{N}\}$ , called generalized weight space,

$V_\lambda = \{v \in V : Hv = \lambda v\}$ , called the eigenspace to  $\lambda$ .

Note that  $V_\lambda$  is a subspace of  $V(\lambda)$  and if  $\lambda$  is not an eigenvalue of  $H$ , then  $V_\lambda = \{0\}$ . Since  $\mathbb{C}$  is algebraically closed, from the Jordan Decomposition Theorem we have:

$$V \cong \bigoplus_{\lambda \in \mathbb{C}} V(\lambda).$$

Set  $W = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda \subset V$  and note that  $W \neq 0$  as  $H$  must have at least one non-zero eigenvalue, hence at least one non-zero eigenvector in  $V$ .

We are interested in the actions of  $E, F$  and  $H$  on  $V(\lambda)$  and  $V_\lambda$ :

**Lemma 3.1.** Let  $\lambda \in \mathbb{C}$ . Then we have:

(a)  $EV(\lambda) \subset V(\lambda + 2)$  and  $EV_\lambda \subset V_{\lambda+2}$ ;

(b)  $FV(\lambda) \subset V(\lambda - 2)$  and  $FV_\lambda \subset V_{\lambda-2}$ ;

(c)  $HV(\lambda) \subset V(\lambda)$  and  $HV_\lambda \subset V_\lambda$ .

*Proof.* (a) For  $v \in V_\lambda$  we have:

$$H(E(v)) = HE(v) = EH(v) + 2E(v) = \lambda E(v) + 2E(v) = (\lambda + 2)E(v),$$

by using the relation  $HE = E(H + 2)$ . This proves the second statement.  
 To prove the first part take  $v \in V(\lambda)$  and let  $k \in \mathbb{N}_0$  be such that  $(H - \lambda)^k v = 0$ .

$$(H - (\lambda + 2))^k (E(v)) = (H - (\lambda + 2))^k (E(v)) = E(H + 2 - (\lambda + 2))^k (v) = E(H - \lambda)^k v = 0,$$

which implies that  $Ev \in V(\lambda + 2)$ .

**(b)** The proof follows a similar idea, using the other relation  $HF = FH - 2F$ .

**(c)** Clearly,  $H(H(v)) = H(\lambda v) = \lambda H v = \lambda^2 v \in V_\lambda$  and  $HV(\lambda) = (H - \lambda)^k (Hv) = (H - \lambda)^k (\lambda v) = \lambda (H - \lambda)^k (v) = 0$ .  $\square$

**Corollary 3.2.** *The space  $W$  is a submodule of  $V$ . In particular,  $W = V$ , if  $V$  is a simple.*

**Remark 3.3.** A consequence of the corollary is that we can improve the decomposition as follows:

$$V \cong \bigoplus_{\lambda \in \mathbb{C}} V_\lambda \quad (3.1)$$

Since  $V$  is finite-dimensional, the decomposition in 3.1 must be finite, i.e.  $V_\lambda \neq 0$  only for finitely many  $\lambda$ .

Fix some  $\mu \in \mathbb{C}$  such that  $V_\mu \neq 0$  and  $V_{\mu+2k} = 0$  for all  $k \in \mathbb{N}$ . With other words, one sees that all the complex numbers appearing in the decomposition must be congruent to one another mod 2. Let  $v \in V_\mu$  be some non-zero element.  $V_{\mu-2k}$  must be zero for some  $k \in \mathbb{N}$  as we saw in the Lemma 3.1 part (b). Therefore, there exists a minimal  $n \in \mathbb{N}$  such that  $F^n v = 0$ . For  $i \in \{1, 2, \dots, n-1\}$  set  $v_i = F^i v$  with  $v_0 = v$ . The decomposition in 3.1 implies that the  $v_i$ 's are linearly independent. Let us denote the subspace, spanned by these vectors, as  $N$ .

**Lemma 3.4.** *We have that  $Ev_0 = 0$  and  $Ev_i = i(\mu - i + 1)v_{i-1}$  for all  $i \in \{1, 2, \dots, n-1\}$ .*

*Proof.* From the Lemma 3.1 part (a), we know that  $Ev_0$  must be an eigenvector of  $H$  and its corresponding value is  $\lambda + 2$ , but the  $\lambda_i$  are ordered the real value, so there is no non-zero eigenvector corresponding to  $\lambda + 2$ . The only possibility is that  $Ev_0$  is zero.

To prove the rest we proceed by induction on  $i$ :

For  $i = 1$ :  $Ev_1 = EFv_0 = FEv_0 + Hv_0 = 0 + \mu v_0 = \mu v_0$ .

Assume that: For  $i > 1$  we have  $Ev_{i-1} = (i-1)(\mu - (i-1) + 1)v_{i-2} = (i-1)(\mu - i + 2)v_{i-2}$ .

Then:  $Ev_i = EFv_{i-1} = FEv_{i-1} + Hv_{i-1} = (i-1)(\mu - i + 2)Fv_{i-2} + (\mu - 2(i-1))v_{i-1} = i(\mu - i + 1)v_{i-1}$ . This proves the claim.  $\square$

**Corollary 3.5.**  *$N$  is a submodule of  $V$ . In particular,  $N = V$  given  $V$  is simple.*

*Proof.*  $N$  is invariant under the actions of  $H$  and  $F$ . The previous Lemma 3.4 provides that it is invariant under the action of  $E$ .  $\square$

**Lemma 3.6.** *We find an exact value for  $\mu$ , namely  $\mu = n - 1$ .*

*Proof.* By the argument used in the proof of Lemma 3.4 we get:

$$EFv_{n-1} = FEv_{n-1} + Hv_{n-1} = n(\mu - n + 1)v_{n-1}.$$

But,  $Fv_{n-1} = 0$  by our assumption, hence we obtain  $n(\mu - n + 1) = 0 \Rightarrow \mu = n - 1$ .  $\square$

Let us summarize the information gained until now regarding the actions of  $E$ ,  $F$  and  $H$  on the basis vectors  $\{v_0, v_1, v_2, \dots, v_{n-1}\}$ :

Actions of $E$ , $F$ and $H$		
$Ev_0 = 0$	$Fv_{n-1} = 0$	$Hv_i = (n - 2i + 1)v_i$
$Ev_i = i(n - i)v_{i-1}$	$F^i v_0 = v_i$	

**Remark 3.7.** With the information we can visualize our results by the following diagram, known as “ladder” diagram:

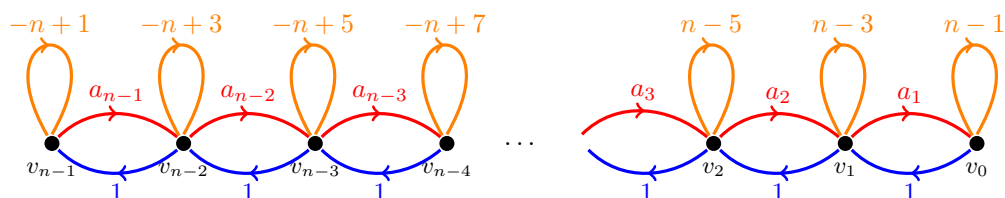


Figure 1: Ladder diagram

Here the red arrows depicts the action of  $E$ , known as raising operator. We set  $a_i = i(n - i)$ . The action of the lowering operator  $F$  is represented in blue.  $H$  acts on itself and is represented by the orange arrows. The labels denote the multiplicities.

**Remark 3.8.** For every  $n \in \mathbb{N}$  the picture above defines on the formal linear span  $N = \{v_0, v_1, \dots, v_{n-1}\}$  the structure of a  $\mathfrak{g}$ -module. Indeed, one has to check that the relations from 2.2 are satisfied for the actions, defined above. This is easy to be done, but there are many identities to be checked. For example, pick the vector  $v_1$ , we want to check the relation  $EF - FE = H$ . Indeed, we obtain  $a_2 - a_1 = n - 3$ , which is exactly what the operator  $H$  does. So, the relation is satisfied. One checks all the other relations for the other vectors in a similar way. We denote this module as  $V^{(n)}$ .

As next, we will consider the main result of today’s talk, namely the classification of the finite-dimensional simple modules.

**Theorem 3.9.** *We have the following statements:*

- (a) For every  $n \in \mathbb{N}$  the module  $V^{(n)}$  is a simple  $\mathfrak{g}$ -module of dimension  $n$ .
- (b) For any  $n, m \in \mathbb{N}$  we have  $V^{(n)} \cong V^{(m)}$ , if and only if  $n = m$ .

(c) Let  $V$  be a simple finite-dimensional  $\mathfrak{g}$ -module of dimension  $n$ . Then  $V \cong V^{(n)}$ .

*Proof.* (a) The module structure of  $V^{(n)}$  follows from Remark 3.8. What is left is to show that this module is simple. Let  $M \subset V^{(n)}$  be a non-zero submodule and  $v \in M, v \neq 0$ . From the Figure 1 we have that  $E^n v = 0$ , in particular  $E^n M = 0$  and hence  $M$  must have a non-trivial intersection with  $\text{Ker}(E)$ . From the same picture it follows that the kernel of  $E$  is the linear span of  $v_0$ , namely it is one-dimensional. Hence  $M$  contains  $v_0$ . Applying the operator  $F$  by induction gives that  $M$  contains all the vectors  $v_i$ . This implies that  $V^{(n)} = M$  and proves the simplicity.

(b) Clear, since the vector spaces of the same dimension are isomorphic.

(c) The last result follows from the Figure 1 above.  $\square$

**Remark 3.10.** One can rescale the basis by setting  $w_i = \frac{1}{i!} v_i$ . Apply the actions of  $E, F$  and  $H$  to the new basis and get the following symmetric picture just by computations:

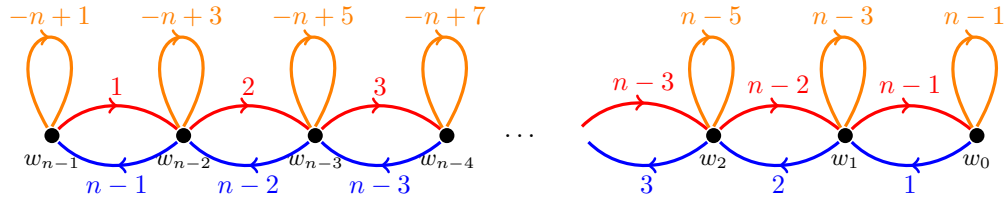


Figure 2: Ladder diagram in the scaled basis.

This picture gives us a way how to construct the matrices for the linear operators  $E, F$  and  $H$ . To get the matrices, apply the linear operators to each of the vectors in the basis  $\{w_0, w_1, \dots, w_{n-1}\}$ .

$$E = \begin{bmatrix} 0 & n-1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & n-2 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & 2 & 0 \\ \vdots & \vdots & \vdots & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & n-2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & n-1 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} n-1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & n-3 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & 5-n & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & 0 & 3-n & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1-n \end{bmatrix}.$$



The final result for this talk is known as Schur's lemma and it shows that we don't have so much freedom when we want to think about intertwiners between finite-dimensional simples.

**Lemma 3.11.** (a) Any non-zero homomorphism between two simple  $\mathfrak{g}$ -modules is an isomorphism.

(b) For any two simple finite-dimensional  $\mathfrak{g}$ -modules  $V$  and  $W$  we have:

$$\text{Hom}_{\mathfrak{g}}(V, W) \cong \begin{cases} \mathbb{C}, & \text{if } V \cong W \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* (a) Let  $\Phi \in \text{Hom}_{\mathfrak{g}}(V, W)$  be some non-zero intertwiner. We shall use the result from Lemma 2.26 saying that  $\text{Ker}(\Phi)$  is a subrepresentation of  $V$  and  $\text{Im}(\Phi)$  is a subrepresentation of  $W$ .

As  $V$  is simple and  $\Phi$  is non-zero, we get that  $\text{Ker}(\Phi) = 0$ , which implies that  $\Phi$  is a monomorphism.

Same in case of  $\text{Im}(\Phi)$ , namely  $W$  is simple and  $\Phi$  is non-zero, which means that  $\text{Im}(\Phi) = W$  and hence,  $\Phi$  is an epimorphism. This implies that  $\Phi$  is an isomorphism.

(b) Part (a) shows that  $\text{Hom}_{\mathfrak{g}}(V, W) = 0$ , if  $V \not\cong W$ .

Assume that  $V \cong W$  and  $\Psi \neq 0 \in \text{Hom}_{\mathfrak{g}}(V, W)$  is another homomorphism. Then we have  $\text{Hom}_{\mathfrak{g}}(V, V) \rightarrow \text{Hom}_{\mathfrak{g}}(V, W)$ , defined by  $\Phi \mapsto \Psi \circ \Phi$ .

We want to show that  $\text{Hom}_{\mathfrak{g}}(V, V) = \mathbb{C}\langle \text{id}_V \rangle \cong \mathbb{C}$ .

If  $\Phi \in \text{Hom}_{\mathfrak{g}}(V, V)$  is non-zero, then it should have a non-zero eigenvalue  $\lambda \in \mathbb{C}$ . Then  $\Phi - \lambda \text{Id}_V \in \text{Hom}_{\mathfrak{g}}(V, V)$ . However, any eigenvector of  $\Phi$  with eigenvalue  $\lambda$  belongs to  $\text{Ker}(\Phi - \lambda \text{Id}_V) \Rightarrow \Phi - \lambda \text{Id}_V$  is not an isomorphism, because we have a non-trivial kernel, so it is not injective. Hence it must be 0. This implies that  $\Phi = \lambda \text{Id}_V$  and this gives  $\text{Hom}_{\mathfrak{g}}(V, V) \cong \mathbb{C}$ . □