Classical Theory II

Reflection groups and Coxeter groups

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So it is the reflection with respect to the hyperplane that is the orthogonal complement of n shifted by γ in the direction of n.

Affine reflection groups

$W < \mathrm{Aff}(V)$ is called an affine reflection group if

- W is generated by affine reflections
- W is *proper*, i.e for any compact sets $K, L \subset V$ the set of $w \in W$ such that $K \cap wL \neq \emptyset$ is finite.

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Lemma:*Every orbit of W is a discrete subset of V with its natural topology*

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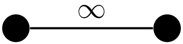
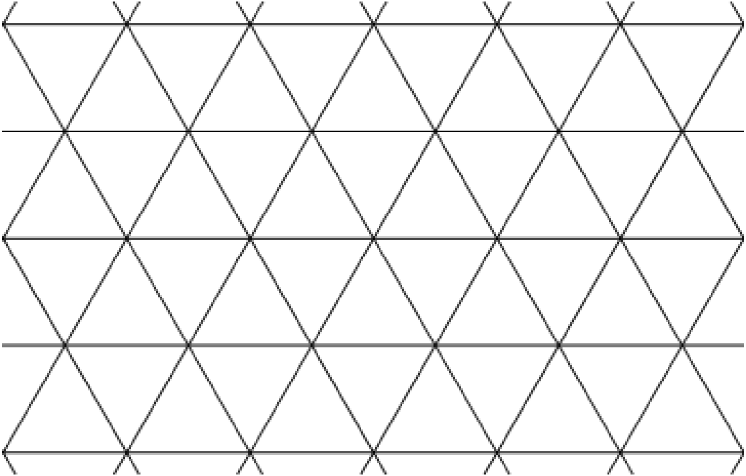


Therefore the infinite dihedral group is a reflection group.

Examples of affine reflection groups

2. Consider $V = \mathbb{R}^2$ with the standard Euclidean structure. Let W denote the affine reflection group generated by the following affine arrangement of hyperplanes



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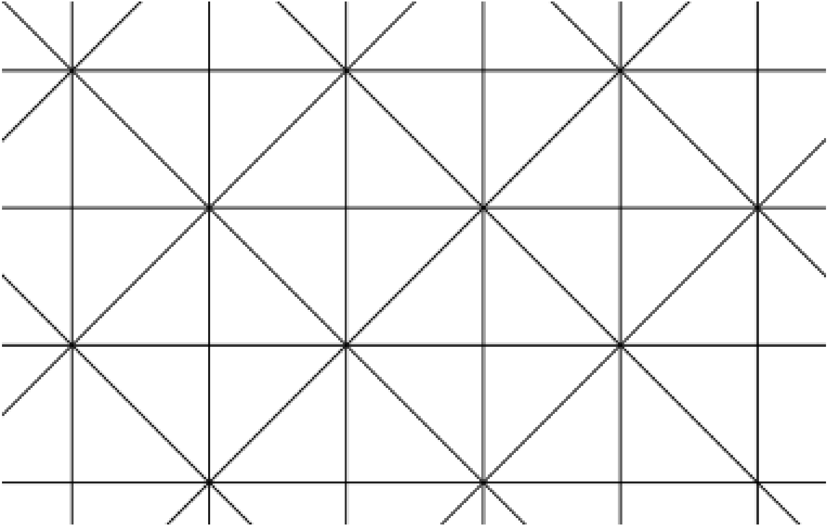

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These two subsets are the connected components of $V \setminus H$ and they are called the *half-spaces* defined by H.If $v, w \in V$ belong to the same half-space of H they are in the same side of H. Otherwise they are on the *opposite sides* and they are *seperated* by H.

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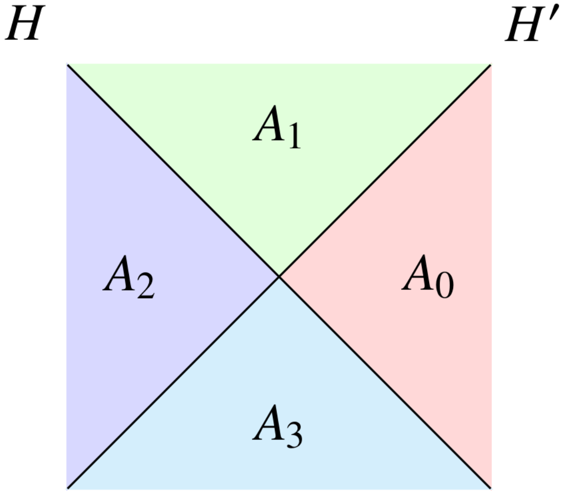
$$V = \bigcup_{\bar{A}\in\bar{\mathcal{A}}} \bar{A}$$

Example



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Note that W acts naturally on the sets $\mathcal{A}$ and $\bar{\mathcal{A}}$.

Some lemmas

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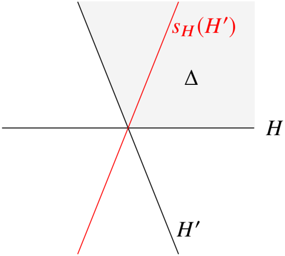
3 *Suppose $H, H' \in \Phi_\Delta$. If H, H' intersect then they do so at an angle $\leq \pi/2$. Moreover the angle is of the form π/m for some $m \in \mathbb{N}$.*

Idea of the proof of 3:



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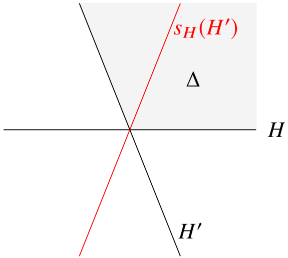

Properness implies there will be finitely many alcoves having the intersection point in their closure. Since reflections preserve angles, there will be $2m$ alcoves with the same angle meeting there. So the angle will be $2\pi/2m = \pi/m$.

Reflection groups are Coxeter groups

For $H, H' \in \Phi_\Delta$, let s, t denote their corresponding reflections. Define

$$m_{st} := \begin{cases} m \text{ (where } \pi/m \text{ is the angle they meet)} & \text{if } H \text{ and } H' \text{ meet} \\ \infty & \text{if } H \text{ and } H' \text{ do not meet} \end{cases}$$

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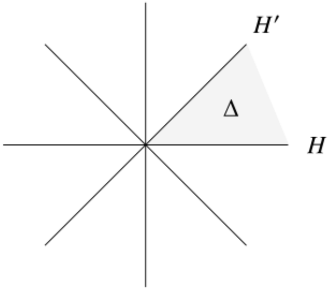

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Theorem: *W admits the following Coxeter presentation:*

$$W = \left\langle s \in S \mid s^2 = \mathrm{id} \textit{ for all } s \in S, (st)^{m_{st}} = \mathrm{id} \textit{ for all distinct } s, t \in S \right\rangle$$

Stroll

A *stroll* is a sequence $\underline{A} := (A_0, A_1, \ldots, A_k)$ of alcoves such that $A_0 = \Delta$ and A_{i-1} and A_i share a face F_i for all $1 \leq i \leq k$ and $A_i \neq A_{i-1}$ for any $i \geq 1$.

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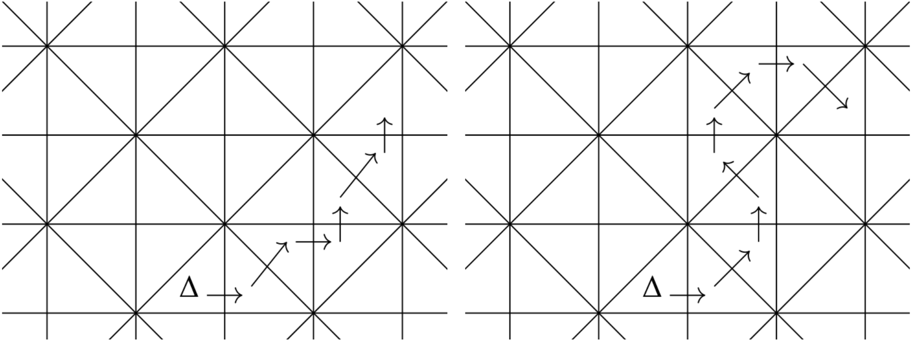
The length of a stroll is the number of times it crosses a hyperplane. A stroll is *reduced* if F_i and F_j are never contained in the same hyperplane for $i \neq j$, i.e if the stroll never passes through the same hyperplane twice.

How a stroll looks like in $\mathbb{R}^2$



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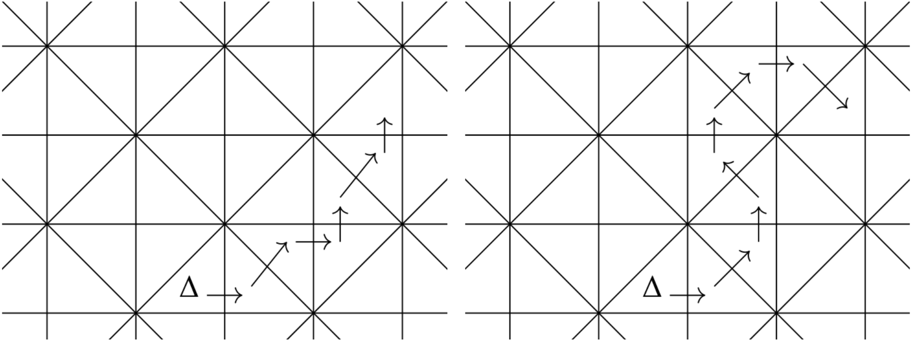

The left one is reduced while the right one is not reduced.

Strolls and expressions

Let $\underline{x} = (s_1, s_2, \ldots, s_k)$ be an expression. Note that Δ and $s_1\Delta$ share a common face (namely, the face that corresponds to s_1). Similarly, Δ and $s_2\Delta$ share a common face. Then $s_1\Delta$ and $s_1 s_2\Delta$ also share a face. Therefore iterating this, to any expression we can associate a stroll

$$\underline{A}(\underline{x}) := A_0 = \Delta, A_1 = s_1\Delta, A_2 = s_1 s_2\Delta, \ldots, A_k = s_1 s_2 \ldots s_k\Delta)$$

Reduced expressions are reduced strolls

Proposition: *An expression $\underline{x}$ for $x \in W$ is reduced if and only if the coressponding stroll $\underline{A}(\underline{x})$ is reduced. Moreover,*

$$\ell(x) = \#\{H \in \Phi \mid H \textit{ seperates } \Delta \textit{ and } x\Delta\}$$

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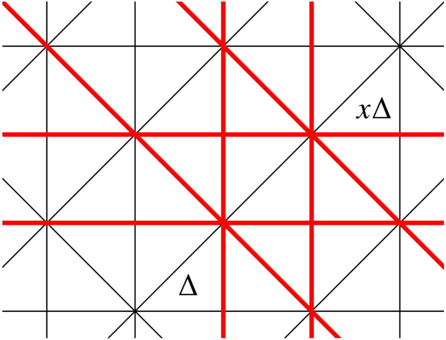

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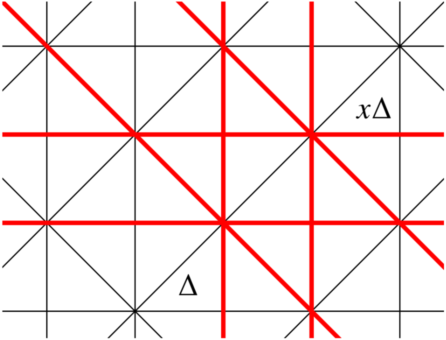

Therefore $\ell(x) = 6$

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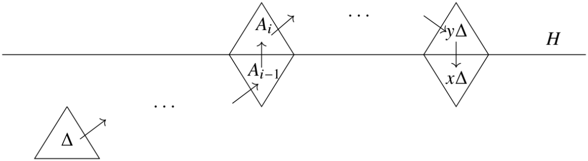
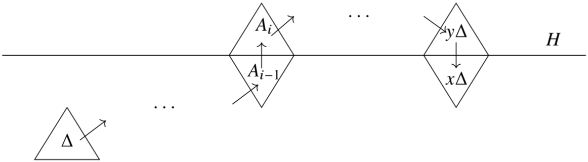
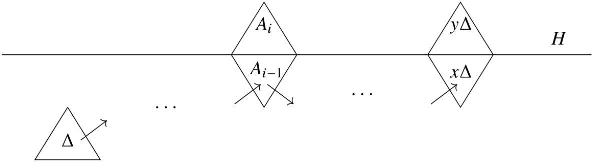




Then $(s_1, \ldots, s_{i-1}, s'_{i+1}, \ldots, s'_{k-1})$ where s'_j are obtained by taking the reflection of the stroll $\underline{A}(\underline{y})$ after the ith step with respect to H (as demonstrated in the figure below) is an expression for x shorter than k.



Matsumoto's theorem

Any two reduced expressions for $x \in W$ may be related by braid relations

This is proven by induction on the length $\ell(x)$. Suppose $\underline{x}_1 = (s_1, \ldots, s_\ell)$,$\underline{x}_2 = (s'_1, \ldots, s'_\ell)$ are two reduced expressions for x. If $s_\ell = s'_\ell$ we are done.

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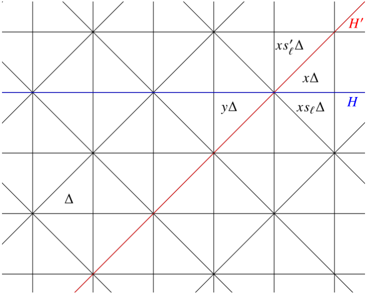

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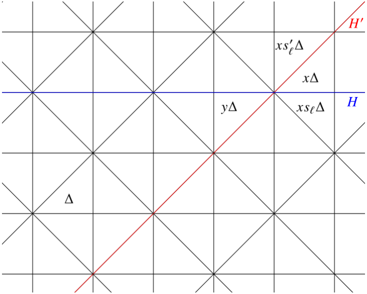

There is a unique alcove which contains $H \cap H'$ in its closure and lies on the same side of H and H' as the fundamental alcove. It is $y\Delta$ for some $y \in W$.

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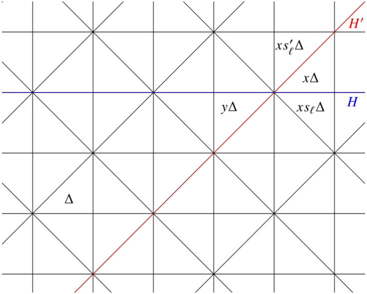

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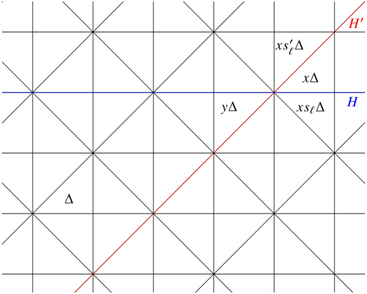

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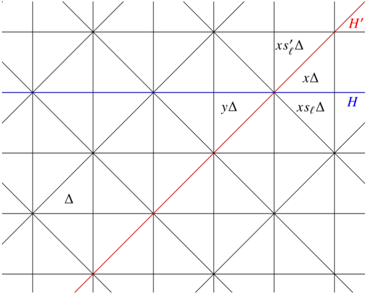

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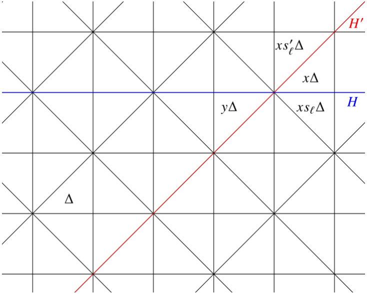

Pick a fixed reduced stroll from Δ to $y\Delta$. It corresponds to a reduced expression $\underline{w} = (t_1, \ldots, t_k)$. Then there is two ways to extend it to $x\Delta$:

$$w_1 := (t_1, \ldots, t_k, \underbrace{s'_\ell, s_\ell, \ldots, s'_\ell, s_\ell}_{m})$$

$$w_2 := (t_1, \ldots, t_k, \underbrace{s_\ell, s'_\ell, \ldots, s_\ell, s'_\ell}_{m})$$

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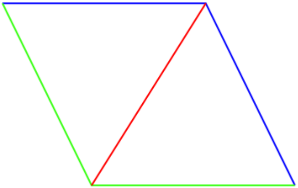
Example ($|S|$=3 and s red):



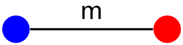
Coxeter complex

Coxeter complex of (W, S) is constructed as follows:

- Take a copy of Δ for each $w \in W$ and call it Δ_w
- For all $w \in W$ and $s \in S$, glue Δ_w to Δ_{ws} via an s-glueing

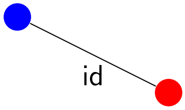
W acts faithfully on the complex by identifying Δ_x with Δ_{wx}.

Coxeter complex of the Dihedral groups

Lets construct the Coxeter complex for: 

Lets say s is the blue one and t the red one. Clearly, an expression here will be an alternating sequence of s, t. And the maximum length of a reduced expression will be m since if it is longer, there will be a sequence of s, t of length m inside it which we can switch up using the braid relation and get rid of either 2 s or 2 t resulting in lowering the length of the expression. Now, lets put this reasoning to pictures.

For $m < \infty$



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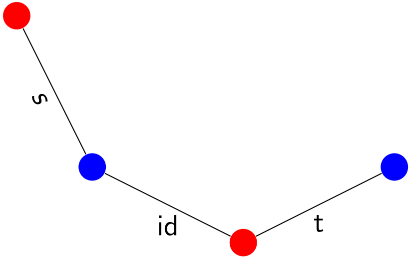

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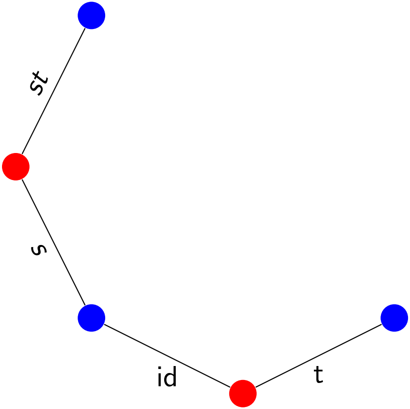

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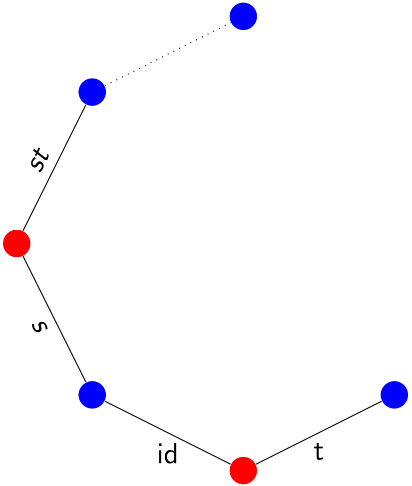

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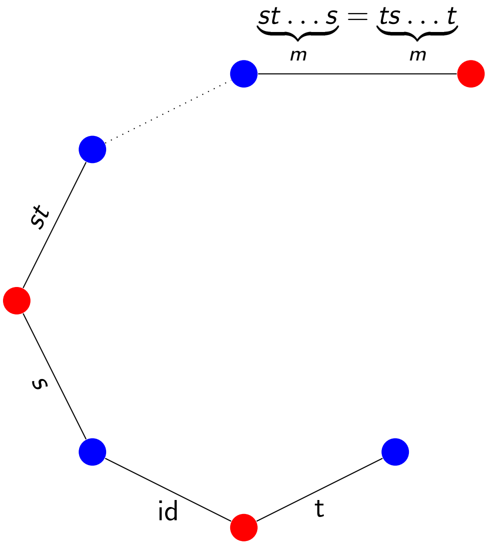

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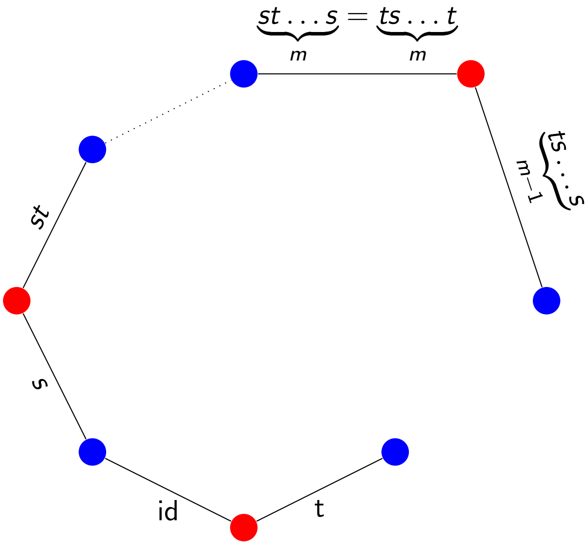

$\underbrace{st\dots s}_{m} = \underbrace{ts\dots t}_{m}$
$\underbrace{ts\dots s}_{m-1}$
st
s
id
t

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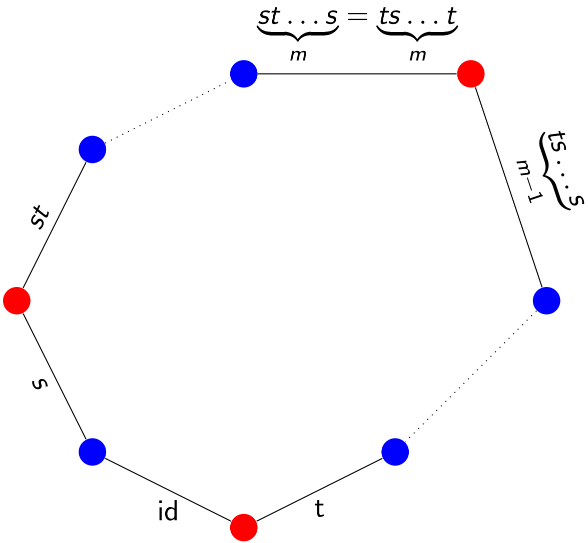

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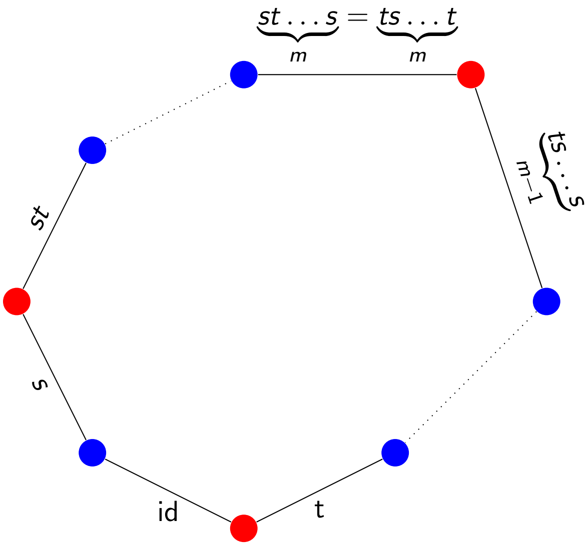
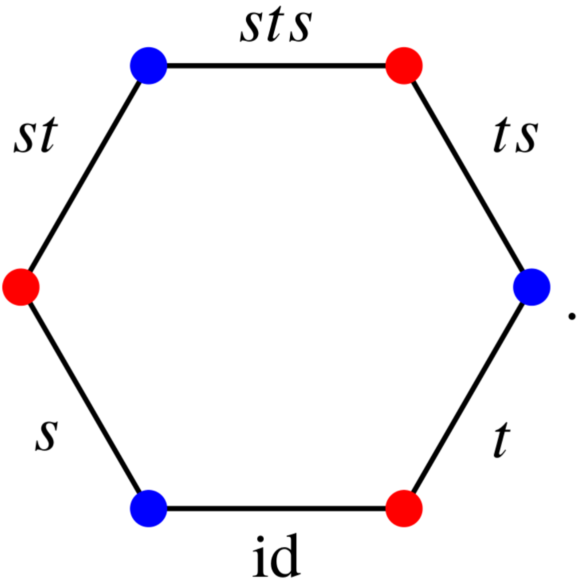
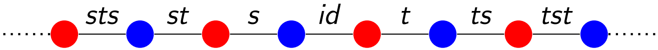


Modulo my bad drawing abilities, this is a 2m-gon.

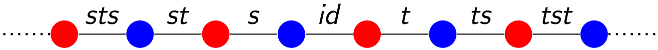
A better drawn example for m=3



m=∞



$m = \infty$



It is an infinite line

Coxeter complex of a Coxeter group W is either homeomorphic to S^n (if $|W|$ is finite of rank $n+1$) or contractible if $|W|$ is infinite.