Lectures on $\mathfrak{sl}_2(\mathbb{C})$ - Seminar talk - The Semisimples

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1 Summary of the results of the first talk used in this talk

Definition 1. A $\mathfrak{g}$-module is a vectorspace V together with three fixed linear operators E, F, and H with satisfy the relations

$$\begin{aligned} EF - FE &= H \\ HE - EH &= 2E \\ HF - FH &= -2F. \end{aligned}$$

This definition is motivated by the fact that the three natural basis vectors of $\mathfrak{g}$ follow the same relations.

Definition 2. A subspace $W \subset V$ is called a $\mathfrak{g}$-submodule provided that it is invariant under the action of the linear operators E, F, and H, i.e.

$$EW \subset W,\ FW \subset W,\ HW \subset W.$$

Any module has two obvious submodules, the zero subspace and the whole space. Any submodule different from the obvious submodule is called a proper submodule.

Definition 3. A module that has no proper submodules is called simple.

Example 4. The kernel and the image of a $\mathfrak{g}$-homomorphism between two modules are both submodules. I.e. let V and W be two modules and $\Phi \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$. Then, $\mathrm{Ker}(\Phi)$ is a submodule of V and $\mathrm{Im}(\Phi)$ is a submodule of W.

Lemma 5. *For any $f \in \mathbb{C}[x]$ (polynomial ring over $\mathbb{C}$) the following holds:*

$$\begin{aligned} f(H)E &= Ef(H+2) \\ f(H)F &= Ff(H-2) \end{aligned}$$

.

Lemma 6. *Let V be a simple finite-dimensional module which contains a non-zero vector v such that $E(v) = 0$ and $H(v) = (n-1)v$. Then, it follows that $V \cong \mathbf{V}^{(n)}$.*

Theorem 7. *(Classification of all simple finite-dimensional modules).*
Any simple finite-dimensional module of dimension n is isomorphic to $\mathbf{V}^{(n)}$.

2 The Semisimples

2.1 Semi-simplicity of finite-dimensional modules

As we have already seen how simple finite-dimensional modules look like, we will proceed to investigate on finite-dimensional modules that are not simple. We will introduce semi-simple modules and we will see that all finite-dimensional modules are at least semi-simple.

Remark 8. Let V and W be two $\mathfrak{g}$-modules. We define the following operators on $V \oplus W$:

$$\begin{aligned} E(v \oplus w) &:= E(v) \oplus E(w) \\ F(v \oplus w) &:= F(v) \oplus F(w) \\ H(v \oplus w) &:= H(v) \oplus H(w) \ \forall v \in V,\ w \in W. \end{aligned}$$

We introduce the notation $nV := \underbrace{V \oplus \cdots \oplus V}_{n \text{ summands}}$.

Proposition 9. *The direct sum of two $\mathfrak{g}$-modules V and W endowed with the operators E, F, and H as given in the remark above is also a $\mathfrak{g}$-module.*

Proof. We have to prove the following:

$$\begin{aligned} (EF - FE)(v \oplus w) &= H(v \oplus w) \\ (HE - EH)(v \oplus w) &= 2E(v \oplus w) \\ (HF - FH)(v \oplus w) &= -2F(v \oplus w) \ \forall v \in V,\ w \in W. \end{aligned}$$

All these equations can be proved by direct calculation.

$$\begin{aligned} (EF - FE)(v \oplus w) &= EF(v \oplus w) - FE(v \oplus w) \\ &= E(F(v) \oplus F(w)) - F(E(v) \oplus E(w)) \\ &= EF(v) \oplus EF(w) - FE(v) \oplus FE(w) \\ &= (EF(v) - FE(v)) \oplus (EF(w) - FE(w)) \\ &= H(v) \oplus H(w) \\ &= H(v \oplus w). \end{aligned}$$

$$
\begin{aligned}
(HE-EH)(v\oplus w) &= HE(v\oplus w)-EH(v\oplus w)\\
&= H(E(v)\oplus E(w))-E(H(v)\oplus H(w))\\
&= HE(v)\oplus HE(w)-EH(v)\oplus EH(w)\\
&= (HE(v)-EH(v))\oplus(HE(w)-EH(w))\\
&= (2E(v))\oplus(2E(w))\\
&= 2(E(v)\oplus E(w))\\
&= 2E(v\oplus w).
\end{aligned}
$$

$$
\begin{aligned}
(HF-FH)(v\oplus w) &= HF(v\oplus w)-FH(v\oplus w)\\
&= H(F(v)\oplus F(w))-F(H(v)\oplus H(w))\\
&= HF(v)\oplus HF(w)-FH(v)\oplus FH(w)\\
&= (HF(v)-FH(v))\oplus(HF(w)-FH(w))\\
&= (-2F(v))\oplus(-2F(w))\\
&= -2(F(v)\oplus F(w))\\
&= -2F(v\oplus w).
\end{aligned}
$$

□

Remark 10. By the very same calculation as above, it is shown that the vectorspace $nV := \underbrace{V\oplus\cdots\oplus V}_{n \text{ summands}}$ endowed with

$$
\begin{aligned}
E(v\oplus\cdots\oplus v) &:= E(v)\oplus\cdots\oplus E(v)\\
F(v\oplus\cdots\oplus v) &:= F(v)\oplus\cdots\oplus F(v)\\
H(v\oplus\cdots\oplus v) &:= H(v)\oplus\cdots\oplus H(v)\ \forall v\in V
\end{aligned}
$$

is also a $\mathfrak{g}$-module.

Next, we will define the following terms: decomposable, indecomposable, and semi-simple modules.

Definition 11. A $\mathfrak{g}$-module V is called *decomposable* if there exist two non-zero $\mathfrak{g}$-modules V_1 and V_2 such that $V \cong V_1 \oplus V_2$. A $\mathfrak{g}$-module which is not decomposable is called *indecomposable.* A $\mathfrak{g}$-module which is isomorphic to a direct sum of (possibly many) simple $\mathfrak{g}$-modules is called *semi-simple.*

Let us recall what we have up to now. A module is called *simple* if there does not exist a proper submodule. A module is called *decomposable* if it is isomorphic to a direct sum of non-zero modules.

A module might be

	simple	not simple
indecomposable		
decomposable		

Generally, there may exist many non-simple but indecomposable modules. I.e. there exist many modules which have a proper submodule but cannot be written as a direct sum. However, the finite case is special. We will see that every finite-dimensional module is at least decomposable into a direct sum of simple submodules (i.e. semi-simple, Weyl's theorem). And - in addition - every indecomposable finite-dimensional module is simple.

To prove Weyl's theorem, we will need the help of a special operator constructed from the "module operators" E, F, and H. Throughout this paragraph, we will assume that V is a finite-dimensional module.

Definition 12. *Casimir operator.* The operator $C := (H+1)^2 + 4FE$ on V is called the Casimir operator.

Lemma 13. *This lemma states some useful relations for the Casimir operator.*
(a) $C = (H-1)^2 + 4EF = H^2 + 1 + 2EF + 2FE$.
(b) $EC = CE$, $FC = CF$, $HC = CH$. *I.e. the Casimir operator commutes with every* $\mathfrak{g}$*-module operator.*

Proof. The proof can be performed by direct calculation. During the proof, we will need commutation relations for polynomial functions of H given in the previous talk.

(a)

$$\begin{aligned} C &\overset{\text{def}}{=} (H+1)^2 + 4FE \\ &\overset{EF-FE=H}{=} H^2 + 2H + 1 + 4(EF - H) \\ &= H^2 - 2H + 1 + 4EF \\ &= (H-1)^2 + 4EF. \end{aligned}$$

$$\begin{aligned} C &\overset{\text{see}above}{=} H^2 - 2H + 1 + 4EF \\ &\overset{EF-FE=H}{=} H^2 - 2(EF - FE) + 1 + 4EF \\ &= H^2 + 1 + 2EF + 2FE. \end{aligned}$$

(b)

$$
\begin{aligned}
HC &= H((H+1)^2+4FE) \\
&= H(H+1)^2+4HFE \\
&\overset{HF=F(H-2)}{=} H(H+1)^2+4F(H-2)E \\
&= (H+1)^2H+4(FHE-2FE) \\
&\overset{HE=E(H+2)}{=} (H+1)^2H+4(FE(H+2)-2FE) \\
&= (H+1)^2H+4FEH \\
&= ((H+1)^2+4FE)H \\
&= CH
\end{aligned}
$$

The equalities $HF = F(H-2)$ and $HE = E(H+2)$ follow from exercise 1.2.2 with $f = 1$.

$$
\begin{aligned}
EC &= E((H+1)^2+4FE) \\
&= E(H+1)^2+4EFE \\
&\overset{(H-1)^2E=E(H+1)^2}{=} (H-1)^2E+4EFE \\
&= ((H-1)^2+4EF)+E \\
&\overset{(a)}{=} CE
\end{aligned}
$$

The equality $(H-1)^2E = E(H+1)^2$ follows again from exercise 1.2.2 with $f(H) = (H-1)^2$. In this case $(H-1)^2E = f(H)E = Ef(H+2) = E(H+2-1)^2 = E(H+1)^2$.

$$
\begin{aligned}
FC &\overset{(a)}{=} F((H-1)^2+4EF) \\
&= F(H-1)^2+4FEF \\
&\overset{(H+1)^2F=F(H-1)^2}{=} (H+1)^2F+4FEF \\
&= ((H+1)^2+4FE)F \\
&= CF
\end{aligned}
$$

The equality $(H+1)^2F = F(H-1)^2$ follows again from exercise 1.2.2 with $f(H) = (H+1)^2$. In this case $(H+1)^2F = f(H)F = Ff(H-2) = F(H-2+1)^2 = F(H-1)^2$. □

Exercise 14. We will need the following result from linear algebra. Let W be a vectorspace, $\lambda \in \mathbb{C}$, $A, B \in \mathrm{End(W)}$ mit $AB = BA$. The eigenspace and the generalized eigenspace with respect to A are given as follows:

$$
\begin{aligned}
W_\lambda &:= \{w \in W | Aw = \lambda w\} \\
W(\lambda) &:= \{w \in W | \exists k \in \mathbb{N} : (A-\lambda)^k w = 0\}
\end{aligned}
$$

Show, that both $W_\lambda$and $W(\lambda)$ are invariant with respect to B.

Solution 15. (a) Let $e \in W_\lambda$. It follows that $Ae = \lambda e \iff BAe = B\lambda e \overset{BA=AB}{\iff} ABe = \lambda Be \iff Be \in W_\lambda$. This proves the first statement.

(b) Let $e \in W(\lambda)$. It follows that $\exists k \in \mathbb{N}$ with

$$
\begin{aligned}
&(A-\lambda)^k e = 0 \\
\iff &B(A-\lambda)^k e = 0 \\
\iff &B(A-\lambda)(A-\lambda)\dots(A-\lambda)e = 0 \\
\iff &(BA - B\lambda)(A-\lambda)...(A-\lambda)e = 0 \\
\iff &(AB - \lambda B)(A-\lambda)\dots(A-\lambda)e = 0 \\
\iff &(A-\lambda)B(A-\lambda)...(A-\lambda)e = 0 \\
\iff &(A-\lambda)\dots(A-\lambda)Be = 0 \\
\iff &(A-\lambda)^k Be = 0
\end{aligned}
$$

It follows that $B \in W(\lambda)$ (with the same $k \in \mathbb{N}$).

Remark 16. Applying the Jordan decomposition theorem with respect to the Casimir operator on the finite-dimensional module V, we obtain

$$V = \bigoplus_{\tau \in \mathbb{C}} V(C, \tau)$$

with the same definition (“generalized eigenspace”) as already given above:

$$V(C,\tau) = \{v \in V | \exists k \in \mathbb{N} : (C-\tau)^k v = 0\}.$$

The Jordan decomposition theorem assumes the above shape since $\mathbb{C}$ is algebraically closed.

The following lemma is needed for the proof of Weyl’s theorem.

Lemma 17. *For any $\tau \in \mathbb{C}$, the subspace $V(C,\tau)$ is a submodule of V. In particular, if V is indecomposable, then exists $\tau \in \mathbb{C}$ with $V = V(C,\tau)$.*

Proof. To demonstrate:

$$
\begin{aligned}
EV(C,\tau) &\subset V(C,\tau) \\
FV(C,\tau) &\subset V(C,\tau) \\
HV(C,\tau) &\subset V(C,\tau)
\end{aligned}
$$

As calculated above, C commutes with the operators E, F, and H. This implies according to exercise 1.3.5 that E, F, and H leave the above subspaces invariant. I.e. all $V(C,\tau)$ are submodules of V. Moreover, if V is indecomposable, there is no proper submodule. I.e. all summands in the Jordan decomposition must vanish except for one. This completes the proof. □

Exercise 18. Show that $C_{\mathbf{V}^{(n)}} = n^2 \mathrm{id}_{\mathbf{V}^{(n)}}$.

Proof. As we are dealing with a multiple of the identiy matrix, we can work with representation matrices of E, F, and H in any basis, since any multiple of the identity matrix looks the same for arbitrary basis vectors. Therefore, we choose to work with the matrix representation of E, F, and H in the scaled basis $(w_0, w_1, \ldots, w_{n-1})$ as given in the previous sub-chapter.

$$H = \begin{bmatrix} n-1 & & & & \\ & n-3 & & & \\ & & \ddots & & \\ & & & n-2n+2 & \\ & & & & n-2n \end{bmatrix}$$

Recall that $C = (H+1)^2 + 4FE$. As H is diagonal, $(H+1)^2$ is also diagonal. The diagonal elements of H are given by $H_{ii} = n - 2(i-1) - 1 = n - 2i + 1$. Thus, the diagonal elements of $(H+1)^2$ are given by $((H+1)^2)_{ii} = (n-2i+2)^2$. From the special form of F and E, it follows that the product FE only contains diagonal elements.

$$\begin{aligned} FE &= \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & n-1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & n-1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 2 & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & & & & \\ & 1(n-1) & & & \\ & & 2(n-2) & & \\ & & & \ddots & \\ & & & & (n-1)1 \end{bmatrix}. \end{aligned}$$

Thus, the diagonal elements of FE are given by $(FE)_{ii} = (i-1)(n-i+1)$. The diagonal elements of C are then given by $(n-2i+2)^2 + 4(i-1)(n-i+1) = n^2 + 4i^2 + 4 - 4ni + 4n - 8i + 4ni - 4i^2 + 4i - 4n + 4i - 4 = n^2$. □

Theorem 19. *Weyl's theorem.*

Every indecomposable finite-dimensional module is simple. Equivalently, every finite-dimensional module is semi-simple.

Remark 20. Sketch of the proof.

1. We calculate the kernel of the operators E and F.
2. Then, we will see that the generalized eigenspaces can only be non-null for the same eigenvalues as already determined for the "true" eigenspaces, i.e. $V(\lambda) \neq 0$ for $\lambda \in \{-\mathrm{n}+1, -\mathrm{n}+3, \ldots, \mathrm{n}-3, \mathrm{n}-1\}$.

3. Then, we will see that the generalized eigenspaces are identical to the "true" eigenspaces, i.e. $V(\lambda) = V_\lambda$.

4. Then, we will be able to construct submodules of V_λ and write down a decomposition into submodules, from which we can finally prove the statement.

Proof. Let V be a non-zero indecomposable finite-dimensional module.

It follows from the above lemma that exists $\tau \in \mathbb{C}$ mit $V = V(C, \tau)$. This can be further specialized since $C_V = n^2 \mathrm{id}_V$ with n being the dimension of the vectorspace V. Therefore:

$$\begin{aligned} V(C,\tau) &= \{v \in V | \exists k \in \mathbb{N} : (C-\tau)^k v = 0\} \\ &= \{v \in V | \exists k \in \mathbb{N} : (n^2-\tau)^k v = 0\} \end{aligned}$$

The only way we can avoid $V(C, \tau)$ from being empty is, if $n^2 = \tau$. Therefore: $V = V(C, n^2)$.

Now, we consider the Jordan decomposition

$$V = \bigoplus_{\lambda \in \mathbb{C}} V(\lambda)$$

with $V(\lambda)$ being the generalized eigenspace with respect to the operator H.

We first claim that E acts injectively on any $V(\lambda)$ except for $\lambda \in \{-n-1, n-1\}$. In other words, the restricted map

$$\begin{aligned} E_{|V(\lambda)} : V(\lambda) &\longrightarrow V \\ v &\longmapsto E_{|V(\lambda)}(v) := E(v) \end{aligned}$$

is injective for all λ except for $\lambda \in \{-n-1, n-1\}$. Since E is a linear operator, we focus on the kernel of E. Let $v \in V(\lambda) \cap \mathrm{Ker}(E)$ and assume that $v \neq 0$.

$$E(H(v)) = (EH)(v) = (HE)(v) - \underbrace{2E(v)}_{=0} = H(E(v)) = 0.$$

I.e. from $v \in V(\lambda) \cap \mathrm{Ker}(E)$ follows $H(v) \in \mathrm{Ker}(E)$. Furthermore, as shown in the previous talk, the operator H leaves $V(\lambda)$ invariant, i.e. $H(v) \in V(\lambda)$. Thus: $H(v) \in V(\lambda) \cap \mathrm{Ker}(E)$. In other words: the space $V(\lambda) \cap \mathrm{Ker}(E)$ is invariant under the action of H.

Furthermore: $V(\lambda) \cap \mathrm{Ker}(E) \neq 0$ since it was assumed that $v \neq 0$. However, then it also follows that $V_\lambda \cap \mathrm{Ker}(E) \neq 0$. This follows like so:

$v \in V(\lambda) \cap \mathrm{Ker}(E) \Rightarrow \exists k \in \mathbb{N} : (H-\lambda)^k v = 0$. Let k be minimal in the sense that there does not exist $k' < k$ with $(H-\lambda)^{k'} v = 0$. Then the element $v' = (H-\lambda)^{k-1} v \neq 0$ and $(H-\lambda)v' = 0$. I.e. $v' \in V_\lambda$. And $v' \in \mathrm{Ker(E)}$ since H leaves the space $V(\lambda) \cap \mathrm{Ker}(E)$ invariant (any application of $H - \lambda$ on a vector from $\mathrm{Ker}(E)$ leaves this vector in $\mathrm{Ker}(E)$).

Let thus $v'' \in V_\lambda \cap \mathrm{Ker}(E)$ and perform the following calculation:

$$\begin{aligned} Cv'' &= ((H+1)^2 + 4FE)v'' \\ &= (H+1)^2 v'' + \underbrace{4FEv''}_{=0} \\ &= (H+1)(H+1)v'' \\ (\text{because } v'' \in V_\lambda) &= (H+1)(\lambda+1)v'' \\ (\text{because } v'' \in V_\lambda) &= (\lambda+1)^2 v''. \end{aligned}$$

At the same time: $Cv'' = n^2 v''$. Thus, it follows that $\lambda = \pm n - 1$. To recall, we have shown, that from $v \neq 0$ follows $\lambda = \pm n - 1$. I.e. by reverting the argument, for any $\lambda \neq \pm n - 1$ it follows that $v = 0$.

In the very same way, it is proved that F acts injectively on any $V(\lambda)$, $\lambda \neq \pm n + 1$.

Furthermore, $V(\lambda) \neq 0$ is only possible if $V_\lambda \neq 0$. In other words: $V_\lambda = 0 \Longrightarrow V(\lambda) = 0$. This can be seen as follows:

$V_\lambda = 0 \Longleftrightarrow \nexists v \in V \setminus \{0\} : (H - \lambda)v = 0$. I.e. any application of $(H - \lambda)$ on v never returns 0. I.e. also $V(\lambda) = 0$.

Thus, $V(\lambda) \neq 0$ only for $\lambda \in \{-n+1, -n+3, \ldots, n-1\}$. We can draw a similar picture for the actions of E and F on the $V(\lambda)$ as already drawn for the V_λ:

$$\ldots 0 \underset{E}{\overset{F}{\leftrightarrows}} V(-n+1) \underset{E}{\overset{F}{\leftrightarrows}} V(-n+3) \underset{E}{\overset{F}{\leftrightarrows}} \ldots \underset{E}{\overset{F}{\leftrightarrows}} V(n-3) \underset{E}{\overset{F}{\leftrightarrows}} V(n-1) \underset{E}{\overset{F}{\leftrightarrows}} 0 \ldots$$

From this picture, it follows that $\mathrm{Ker}(E) = V(n-1)$ and $\mathrm{Ker}(F) = V(-n-1)$ as the kernels for the other subspaces are zero (1) by the above statements and (2) by the fact that the kernel of a linear operator that acts on a vector space with the zero element only must be necessarily zero.

It follows that all vector spaces $V(-n+1), V(-n+3), \ldots, V(n-1)$ have the same dimension. To see this, consider the fact that every injective linear map to itself is an isomorphism. The map FE is an injective linear map to itself and thus an isomorphism. This is only possible if the involved vector spaces have the same dimension.

We will next show that $V_\lambda = V(\lambda)$ for $\lambda \in \{-n+1, -n+3, \ldots, n-1\}$. (This does not hold in general, this only holds since these vector spaces are (generalized) eigenspaces with respect to the linear operator H.

For this, define A_i be the restriction of F^i to $V(n-1)$. A_i is then an isomorphism. Set $A = A_{n-1}$. I.e. A maps from $V(n-1)$ to $V(-n+1)$.

Let C_1 and H_1 be the restrictions of C and H on $V(n-1)$ and C_2 and H_2 be the restrictions of C and H on $V(-n+1)$. As $C_1 = (n-1)^2\mathrm{id}$ and $C_2 = (-n+1)^2\mathrm{id}$ it follows that

$$AC_1 = C_2 A.$$

And using $FH = (H+2)F$ (standard relation of any H-operator in a module) multiple times we get:

$$\begin{aligned}
AH_1 &= F^{n-1}H = F^{n-2}FH = F^{n-2}(H+2)F = F^{n-2}HF + 2F^{n-1} \\
&= F^{n-3}FHF + 2F^{n-1} = F^{n-3}(H+2)F^2 + 2F^{n-1} = F^{n-3}HF^2 + 4F^{n-1} \\
&= \cdots \\
&= HF^{n-1} + 2(n-1)F^{n-1} = (H_2 + 2(n-1))A.
\end{aligned}$$

As $\mathrm{Ker}(E) = V(n-1)$ and $C = (H+1)^2 + 4FE$ we have

$$C_1 = (H_1 + 1)^2$$

As $\mathrm{Ker}(F) = V(-n+1)$ and $C = (H-1)^2 + 4EF$ we have

$$C_2 = (H_2 - 1)^2.$$

Thus, we have:

$$\begin{array}{rcl}
(H_1+1)^2 & = & C_1 \\
 & = & A^{-1}AC_1 \\
 & = & A^{-1}C_2A \\
 & = & A^{-1}(H_2-1)^2A \\
\text{(see below)} & = & A^{-1}A(H_1+1-2n)^2 \\
 & = & (H_1+1-2n)^2
\end{array}$$

The proof of partial step from above works as follows: from $AH_1 = (H_2 - 2(n-1))A = H_2A + 2(n-1)A = H_2A + 2n - 2A = H_2A - A + 2n - A = (H_2-1)A + 2n - A \iff (H_2-1)A = A(H_1+1-2n)$. Thus: $(H_2-1)^2A = (H_2-1)A(H_1+1-2n) = A(H_1+1-2n)^2$.

Summarizing the above, we have

$$\begin{array}{rcl}
(H_1+1)^2 & = & (H_1+1-2n)^2 \\
H_1^2 + 2H_1 + 1 & = & H_1^2 + 2(1-2n)H_1 + (1-2n)^2 \\
2H_1 + 1 & = & 2(1-2n)H_1 + (1-2n)^2 \\
2H_1 + 1 & = & 2H_1 - 4nH_1 + 1 - 4n + 4n^2 \\
4nH_1 & = & 4n^2 - 4n \\
H_1 & = & n-1
\end{array}$$

This in turn implies that $V(n-1) = V_{n-1}$. This follows like so: The inclusion $V_{n-1} \subset V(n-1)$ holds in any case. Therefore, it only remains to show that $V(n-1) \subset V_{n-1}$. Let $v \in V(n-1)$. Then: $Hv = (n-1)v \Rightarrow v \in V_{n-1}$ since $V_{n-1} = \{v \in V | (H-(n-1))v = 0\} = \{v \in V | Hv = (n-1)v\}$.

Furthermore $A_i H = (H+2i)A_i$. This follows like so:

$$\begin{aligned}
A_i H &= \underbrace{F\cdots F}_{i \text{ times}} H \\
&\overset{FH=(H+2)F}{=} \underbrace{F\cdots F}_{i-1 \text{ times}} (H+2)F \\
&= \underbrace{F\cdots F}_{i-1 \text{ times}} HF + 2A_i \\
&= \underbrace{F\cdots F}_{i-2 \text{ times}} (H+2)FF + 2A_i \\
&= \underbrace{F\cdots F}_{i-2 \text{ times}} HFF + 4A_i \\
&= \ldots \\
&= HA_i + 2iA_i \\
&= (H+2i)A_i.
\end{aligned}$$

In addition: $v \in V_{n-1} \implies A_1 v \in V_{n-3}$. This is already clear from the definition of A_i. However, we can calulcate this explicitly as follows: $v \in V_{n-1} \Rightarrow Hv = (n-1)v \Rightarrow A_1 Hv = (n-1)A_1 v \Rightarrow (H+2)A_1 v = HA_1 v + 2A_1 v = (n-1)A_1 v \Rightarrow HA_1 v = (n-3)A_1 v \Rightarrow A_1 v \in V_{n-3}$. In addition: A_1 is an isomorphism from $V(n-1)$ to $V(n-3)$. It follows that $V(n-3) = V_{n-3}$. By iterative argumentation, it follows that $V_\lambda = V(\lambda)$ for $\lambda \in \{-n+1, -n+3, \ldots, n-1\}$.

Let $\{v_1, \ldots, v_k\}$ be a basis of V_{n-1}. For $i \in \{1, \ldots, k\}$ denote by W_i the linear span of $\{v_i, Fv_i, \ldots, F^{n-1}v_i\}$. I.e.

$$\begin{aligned}
W_1 &= \operatorname{span}\{v_1, Fv_1, \ldots, F^{n-1}v_1\} \\
W_2 &= \operatorname{span}\{v_2, Fv_2, \ldots, F^{n-1}v_2\} \\
&\ldots \\
W_k &= \operatorname{span}\{v_k, Fv_k, \ldots, F^{n-1}v_k\}
\end{aligned}$$

It follows that $V \cong W_1 \oplus \cdots \oplus W_k$. From the first seminar talk, every W_i is a submodule of V. As V was assumed to be indecomposable, it follows that $k = 1$ and $\dim(V_{n-1}) = 1$. Furthermore, $V \cong \operatorname{span}\{v_1, Fv_1, \ldots, F^{n-1}v_1\} = \mathbf{V}^{(n)}$ which was already shown to be simple. We have thus proven the first part of Weyl's Theorem: every indecomposable finite-dimensional module is simple. If the assumption that V is simple does not hold, we have shown that $V \cong W_1 \oplus \cdots \oplus W_k$ with every W_i being simple, thus V is at least semisimple. $\square$

2.2 Tensor products of finite-dimensional modules

Tensor product representations occur in physics when it coms to the rules for constructing the possible total spin of a system consisting of two subsystems with spin j_1 and j_2. For two irreducible representations $D^{(j_1)}$ and $D^{(j_2)}$,we get the decompoition of the tensor product representation into direct sums of irreducible representations as follows (Clebsch-Gordan series):

$$D^{(j_1)} \otimes D^{(j_2)} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D^{(j)}.$$

Definition 21. *Tensor product of two* modules. Let V and W be two modules. The operators in the tensor product space are defined as

$$\begin{aligned} E(v \otimes w) &= E(v) \otimes w + v \otimes E(w) \\ F(v \otimes w) &= F(v) \otimes w + v \otimes F(w) \\ H(v \otimes w) &= H(v) \otimes w + v \otimes H(w) \end{aligned}$$

Exercise 22. Show that with the above definitions of the operators, the tensor product space is indeed a module.

Solution 23. We have to prove the following:

$$\begin{aligned} (EF - FE)(v \otimes w) &= H(v \otimes w) \\ (HE - EH)(v \otimes w) &= 2E(v \otimes w) \\ (HF - FH)(v \otimes w) &= -2F(v \otimes w) \end{aligned}$$

The proof follows by direct calculation:

$$\begin{aligned} (EF - FE)(v \otimes w) &= EF(v \otimes w) - FE(v \otimes w) \\ &= E(F(v) \otimes w + v \otimes F(w)) - F(E(v) \otimes w + v \otimes E(w)) \\ &= EF(v) \otimes w + F(v) \otimes E(w) + E(v) \otimes F(w) + v \otimes EF(w) \\ &\quad -FE(v) \otimes w - E(v) \otimes F(w) - F(v) \otimes E(w) - v \otimes FE(w) \\ &= EF(v) \otimes w - FE(v) \otimes w + v \otimes EF(w) - v \otimes FE(w) \\ &= (EF(v) - FE(v)) \otimes w + v \otimes (EF(w) - FE(w)) \\ &= H(v) \otimes w + v \otimes (H(w)) \\ &= H(v \otimes w) \end{aligned}$$

The other relations are proved in the same way.

Definition 24. Let $n \in \mathbb{N}$. We denote by $V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n \text{ factors}}$.

Exercise 25. Let V and W be two modules. Check that the map

$$\begin{aligned} \Phi : \quad V \otimes W &\longrightarrow \quad W \otimes V \\ (v \otimes w) &\longmapsto \quad \Phi(v \otimes w) := w \otimes v \end{aligned}$$

is an isomorphism.

Solution 26. It is to be shown that Φ is injective and surjective.

For the injective part, we prove that $\mathrm{Ker}(\Phi) = \{0\}$. This can be done since Φ is linear. The zero element of a tensor product space is the tensor product of the individual zero elements. Let $y \in W \otimes V = 0 = 0_W \otimes 0_V$. Then $\Phi(x) = \Phi(v \otimes w) = w \otimes v = 0 \Longleftrightarrow v = 0,\ w = 0 \Longrightarrow x = 0$.

The surjective part is trivial: Let $y = w \otimes v \in W \otimes V$. Then $x := v \otimes w$ satisfies the relation $\Phi(x) = y$.

Exercise 27. Let $V_1,\ V_2,\ W$ be modules. Prove that $(V_1 \oplus V_2) \otimes W \simeq V_1 \otimes W \oplus V_2 \otimes W$.

Solution 28. We work with the decomposition of $v_1 \in V_1,\ v_2 \in V_2,\ w \in W$ into basis elements. Let $V_1 = \mathrm{span}\{e_{1i}\}_{i=1,\dots m_1},\ V_2 = \mathrm{span}\{e_{2j}\}_{j=1,\dots m_2},\ W = \mathrm{span}\{f_k\}_{k=1,\dots n}$ and $V_1 \ni v_1 = \sum_i a_{1i} e_{1i},\ V_2 \ni v_2 = \sum_j a_{2j} e_{2j},\ W \ni w = \sum_k b_k f_k$. Then

$$\begin{aligned}
(v_1 \oplus v_2) \otimes w &= \left[\sum_i a_{1i}(e_{1i},0) + \sum_j a_{2j}(0,e_{2j})\right] \otimes \sum_k b_k f_k \\
&= \sum_k b_k \left[\sum_i a_{1i}(e_{1i},0) + \sum_j a_{2j}(0,e_{2j})\right] \otimes f_k \\
&= \sum_k b_k \left[\sum_i a_{1i}(e_{1i},0) \otimes f_k + \sum_j a_{2j}(0,e_{2j}) \otimes f_k\right] \\
&= \sum_i a_{1i}(e_{1i},0) \otimes \sum_k b_k f_k + \sum_j a_{2j}(0,e_{2j}) \otimes \sum_k b_k f_k \\
&\simeq \left(\sum_i a_{1i} e_{1i} \otimes \sum_k b_k f_k, \sum_j a_{2j} e_{2j} \otimes \sum_k b_k f_k\right) \\
&= (v_1 \otimes w, v_2 \otimes w)
\end{aligned}$$

Exercise 29. Let $U,\ V,\ W$ be modules. Prove that $U \otimes (V \otimes W) = (U \otimes V) \otimes W$.

Theorem 30. *Let $m,\ n \in \mathbb{N}$ such that $m \leq n$. Then*

$$\mathbf{V}^{(n)} \otimes \mathbf{V}^{(m)} \simeq \mathbf{V}^{(n-m+1)} \oplus \mathbf{V}^{(n-m+3)} \oplus \cdots \oplus \mathbf{V}^{(n+m-3)} \oplus \mathbf{V}^{(n+m-1)}.$$

Proof. We prove the theorem by induction on m.

Let $m = 1$. To be verified: $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(1)} \simeq \mathbf{V}^{(n)}$. Observe that $\mathbf{V}^{(1)} \simeq \mathbb{C}$. Let $\{v_i\}_{i=1,\dots n}$ be a basis in $\mathbf{V}^{(n)}$ and $\mathbf{1}$ be a basis in $\mathbb{C}$. Define:

$$\begin{aligned}
\iota : \mathbf{V}^{(n)} \otimes \mathbb{C} \quad &\longrightarrow \quad \mathbf{V}^{(n)} \\
x \quad &\longmapsto \quad \iota(x) = \iota(\sum_{i=1}^{n} e_i \lambda v_i \otimes \mathbf{1}) := \lambda \sum_{i=1}^{n} e_i v_i.
\end{aligned}$$

It is obvious that ι is an isomorphism.

Let $m = 2$. To be verified: $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)} \simeq \mathbf{V}^{(n-1)} \oplus \mathbf{V}^{(n+1)}$. Observe that $\mathbf{V}^{(2)} \simeq \mathbb{C}^2$. Let $e_1,\ e_2$ be the natural basis of $\mathbb{C}^2$. I.e.:

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Observe that with the operators $e,\ f$, and h, the following relations hold:

$$\begin{aligned}
ee_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \\
ee_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1 \\
fe_1 &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2 \\
fe_2 &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \\
he_1 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1 \\
he_2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -e_2
\end{aligned}$$

Assume that $\mathbf{V}^{(n)} = \operatorname{span}\{v_i\}_{i=1}^n$ as already given above. Now, for $v_0 \otimes e_1 \in \mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)}$ calculate

$$\begin{aligned}
&E(v_0 \otimes e_1) = Ev_0 \otimes e_1 + v_0 \otimes ee_1 = 0 \otimes e_1 + v_0 \otimes 0 = 0 \\
&H(v_0 \otimes e_1) = Hv_0 \otimes e_1 + v_0 \otimes he_1 = (n-1)v_0 \otimes e_1 + v_0 \otimes e_1 = nv_0 \otimes e_1
\end{aligned}$$

I.e. (according to exercise 1.2.11) $\mathbf{V}^{(n+1)}$ is a direct summand of $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)}$.

By the same reasoning, define $w := v_1 \otimes e_1 - (n-1)v_0 \otimes e_2$. Calculate

$$\begin{aligned}
E(w) &= \cdots = 0 \\
H(w) &= \cdots = (n-2)w
\end{aligned}$$

I.e. (again according to exercise 1.2.11) $\mathbf{V}^{(n-1)}$ is a direct summand of $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)}$. There are no more subspaces because the dimension of the space spanned by the two subspaces is already $2n$. I.e. $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(2)} \simeq \mathbf{V}^{(n-1)} \oplus \mathbf{V}^{(n+1)}$.

We prove now the induction step, i.e. we assume that the decomposition I.e. we show that from the assumption that the decomposition

$$\mathbf{V}^{(n)} \otimes \mathbf{V}^{(m)} = \mathbf{V}^{(n-m+1)} \oplus \mathbf{V}^{(n-m+3)} \oplus \cdots \oplus \mathbf{V}^{(n+m-1)}$$

holds for $m \in 1, \ldots, k-1$ follows that the decomposition also holds for $m = k$.

We do this as follows: we compute $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k-1)} \otimes \mathbf{V}^{(2)}$ in two different ways (using the "associativity" of the tensor product).

$$\begin{aligned}
\mathbf{V}^{(n)} \otimes \left(\mathbf{V}^{(k-1)} \otimes \mathbf{V}^{(2)}\right) &= \mathbf{V}^{(n)} \otimes \left(\mathbf{V}^{(k)} \oplus \mathbf{V}^{(k-2)}\right) \\
&= \left(\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k)}\right) \oplus \left(\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k-2)}\right) \\
&= \left(\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k)}\right) \oplus \left(\mathbf{V}^{(n-k+3)} \oplus \mathbf{V}^{(n-k+5)} \oplus \cdots \oplus \mathbf{V}^{(n+k-5)} \oplus \mathbf{V}^{(n+k-3)}\right)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\left(\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k-1)}\right) \otimes \mathbf{V}^{(2)} &= \left(\mathbf{V}^{(n-k+2)} \oplus \mathbf{V}^{(n-k+4)} \oplus \cdots \oplus \mathbf{V}^{(n+k-4)} \oplus \mathbf{V}^{(n+k-2)}\right) \otimes \mathbf{V}^{(2)} \\
&= \left(\bigoplus_{i=0}^{k-2} \mathbf{V}^{(n-k+2+2i)}\right) \otimes \mathbf{V}^{(2)} \\
&= \bigoplus_{i=0}^{k-2} \left(\mathbf{V}^{(n-k+3+2i)} \oplus \mathbf{V}^{(n-k+1+2i)}\right) \\
&= \mathbf{V}^{(n-k+1)} \oplus \mathbf{V}^{(n-k+3)} \oplus \cdots \oplus \mathbf{V}^{(n+k-3)} \oplus \mathbf{V}^{(n+k-1)} \\
&\quad\ \mathbf{V}^{(n-k+3)} \oplus \mathbf{V}^{(n-k+5)} \oplus \cdots \oplus \mathbf{V}^{(n+k-3)} \oplus \mathbf{V}^{(n+k-1)}
\end{aligned}$$

Comparing these two results, we get:

$$\mathbf{V}^{(n)} \otimes \mathbf{V}^{(k)} = \mathbf{V}^{(n-k+1)} \oplus \mathbf{V}^{(n-k+3)} \oplus \cdots \oplus \mathbf{V}^{(n+k-3)} \oplus \mathbf{V}^{(n+k-1)}$$

□