Diagrammatic algebra: a prototypical example

Or: Mind your diagrams

Daniel Tubbenhauer

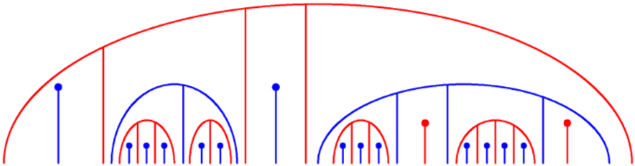

Figure: A light leaf. (Picture from https://arxiv.org/pdf/1702.00039.pdf.)

Seminar Diagrammatic algebra: a prototypical example (Seminar MAT572)

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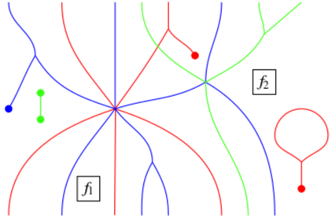

Figure: Soergel diagrams. (Picture from the course book.)

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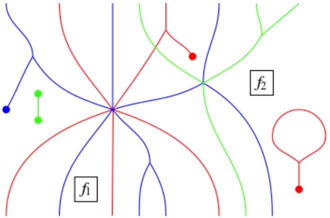

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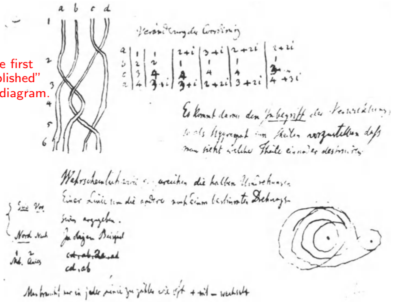
The topic of the seminar: the classical part

The first
“published”
braid diagram.



(Page 283 from Gauß’ handwritten notes, volume seven, $\leq$1830).

Let Γ be a Coxeter graph, *i.e.* a graph with labeled edges.

Artin ∼1925, Tits ∼1961++. The (Gauß–)Artin–Tits group and its Coxeter group quotient are given by generators-relations:

$$\mathrm{AT}(\Gamma) = \langle \beta_i \mid \underbrace{\cdots \beta_i\beta_j\beta_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots \beta_j\beta_j\beta_j}_{m_{ij} \text{ factors}} \rangle$$

$$\downarrow$$

$$\mathrm{W}(\Gamma) = \langle s_i \mid s_i^2 = 1, \underbrace{\cdots s_is_js_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots s_js_is_j}_{m_{ij} \text{ factors}} \rangle$$

This innocent looking definition generalize ▸ Click! and polyhedron groups.

Why Coxeter groups?

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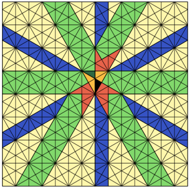

Figure: Kazhdan–Lusztig cells in affine type G_2. (Picture from http://www.maths.usyd.edu.au/u/guilhot/.)

- They appear in many different contexts in mathematics: platonic solids, semisimple Lie algebras, algebraic groups, finite simple groups, quivers, cluster algebras, singularities of hypersurfaces...
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Algebras lend themselves to pictorial presentations

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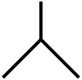
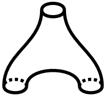
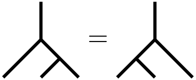
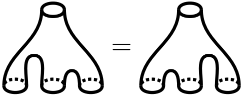
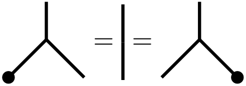
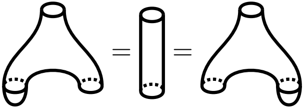
multiplication		
unit		
associativity		
unitality		

Coalgebras lend themselves to pictorial presentations

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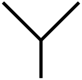
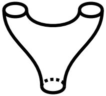
comultiplication		
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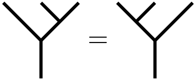
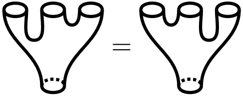
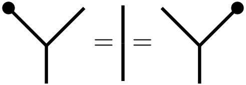
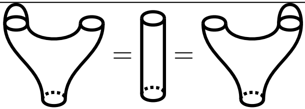
coassociativity		
counitality		

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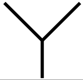
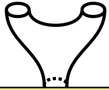
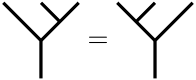
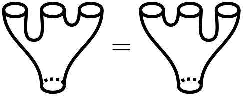
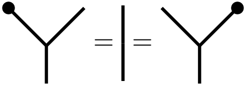
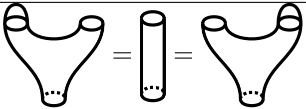
comultiplication		
coassociativity		
counitality		

Main observation.

Use diagrammatic and topological reasoning to tackle problems from algebra!

Soergel diagrams a.k.a. categorical Coxeter groups.

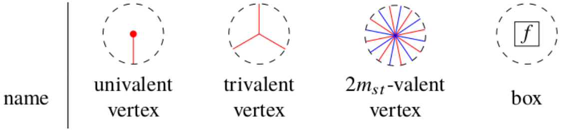
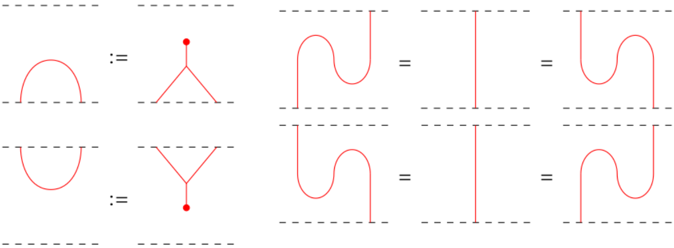


Figure: The generating Soergel diagrams. (Picture from the course book.)



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Main point.

The innocent looking diagrams describe rich mathematics.

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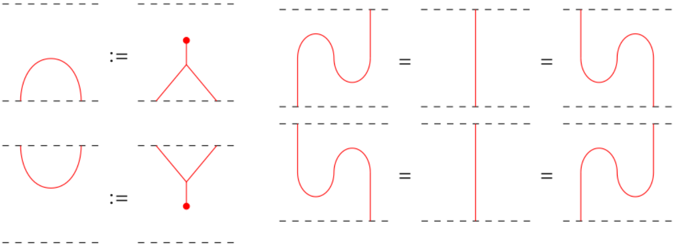

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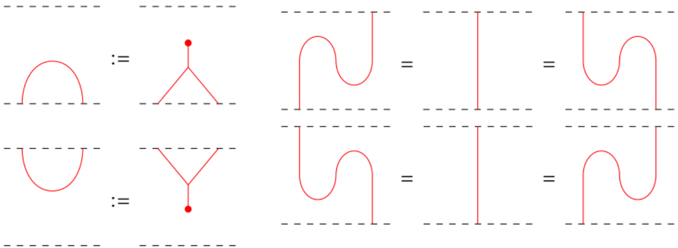
To learn "How" is what this seminar is all about.



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Why Soergel diagrams?

- Soergel bimodules were (and are) in the heart of an explosion of new discoveries in representation theory, algebraic combinatorics, algebraic geometry and knot theory.
- They have amazing associated combinatorics, *e.g.* fractal structures in prime characteristic:

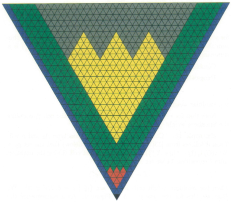

Figure: $p$cells in affine type A_2. (Picture from "H.H. Andersen, Cells in affine Weyl groups and tilting modules".)

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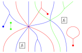

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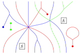

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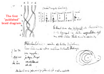

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Coxeter groups come from geometry and arise from reflecting in hyperplanes:

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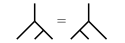
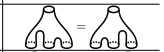
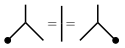
unit		
associativity		
unitality		

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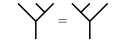
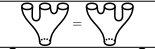
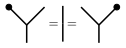
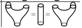
coassociativity		
counitality		

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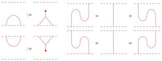

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There is still much to do...

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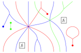

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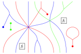

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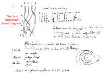

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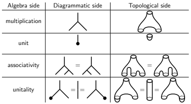
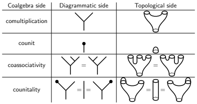


Coalgebras lend themselves to pictorial presentations



Soergel diagrams a.k.a. categorical Coxeter groups.

Figure: The generating Soergel diagrams. (Picture from the course book.)

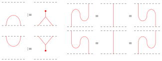


Figure: Some (topological) relations. (Pictures from the course book.)

Thanks for your attention!

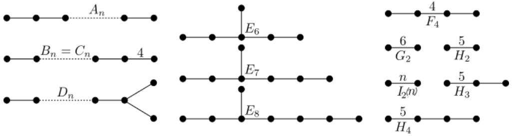
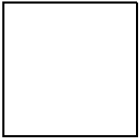


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

Type A_3 $\leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S_4.
Type B_3 $\leftrightsquigarrow$ cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.
Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.
For $I_2(4)$ we have a 4-gon:

Idea (Coxeter ∼1934++).



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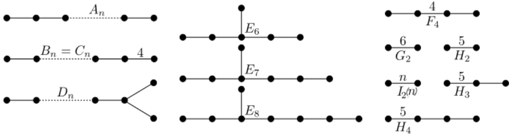


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

Type $A_3 \leftrightsquigarrow$ tetra

Fact. The symmetries are given by exchanging flags.

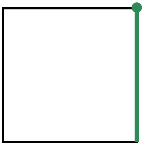
Type $B_3 \leftrightsquigarrow$ cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.

Type $H_3 \leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Fix a flag F.

Idea (Coxeter ∼1934++).



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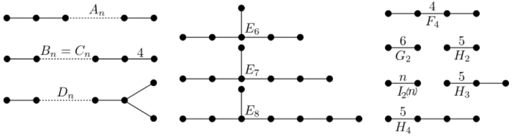

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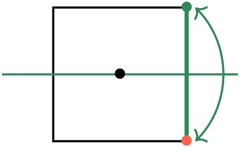
For $I_2(4)$ we have a 4-gon:

Fix a flag F.

Idea (Coxeter ∼1934++).

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.



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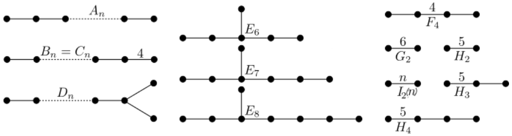

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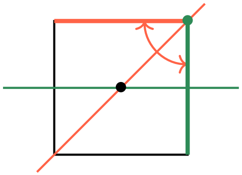
For $\mathrm{I}_2(4)$ we have a 4-gon:

Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Idea (Coxeter ∼1934++).



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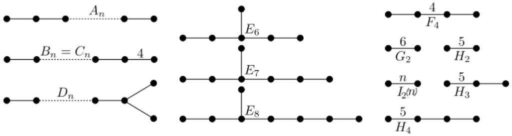

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For $\mathrm{I}_2(4)$ we have a 4-gon:

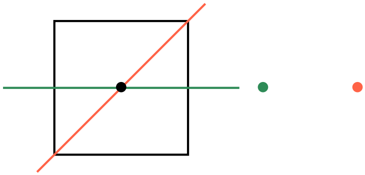
Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Idea (Coxeter ∼1934++).



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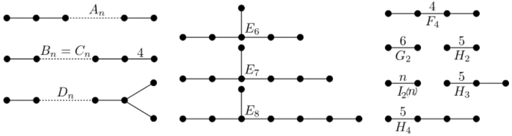


Figure: The Coxeter graphs of finite type. (Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Examples.

This gives a generator-relation presentation.

Type A_3 $\leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S_4.

And the braid relation measures the angle between hyperplanes.

Type B_3 $\leftrightsquigarrow$

Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Fix a flag F.

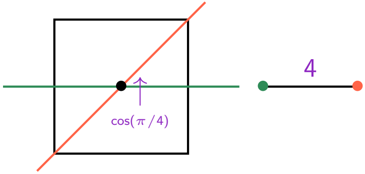
Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Connect i, j by an n-edge for H_i, H_j having angle $\cos(\pi/n)$.

Idea (Coxeter ∼1934++).



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Coxeter groups come from geometry and arise from reflecting in hyperplanes:

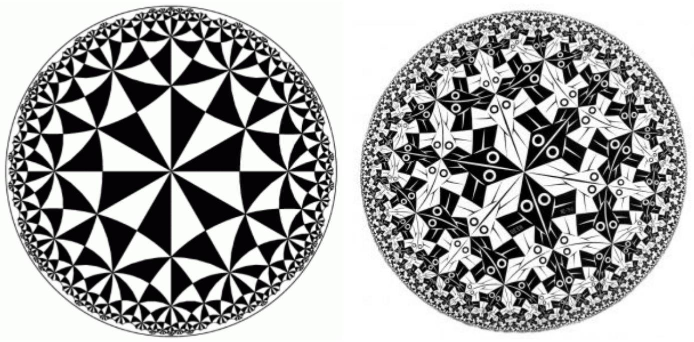

Figure: Coxeter's illustration of a hyperbolic Coxeter group (left) and Escher's version (right). (Picture from https://brewminate.com/escher-and-coxeter-a-mathematical-conversation/.)

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