The diagrammatic beauty of $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_n))$: Part I

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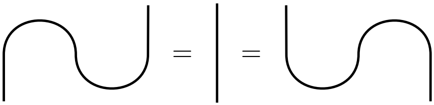
The uncategorified story

March 2014



1. The diagrammatic calculus
   - $\mathfrak{sl}_2$-webs
   - The $\mathfrak{sl}_2$-flow lines
   - Algebra is rigid

2. The representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$
   - The algebra
   - Connection to the diagrammatic calculus
   - Invariant tensors

3. Connection to the $\mathfrak{sl}_n$-link-polynomials
   - The Jones polynomial
   - Reshetikhin-Turaev: Jones is an intertwiner
   - RT polynomials using $\mathfrak{sl}_n$-webs

An old story: Rumer, Teller and Weyl (1932)

2. Fundamentalsatz: Alle linearen Abhängigkeiten zwischen den Monomen ergeben sich (in einem algebraisch genauer präzisierten Sinne) aus der einen Identität (2).

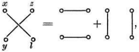
Wir werden uns hier auf den ersten, nicht aber auf den zweiten Fundamentalsatz stützen; vielmehr wird durch unsere Überlegungen ein neuer Beweis des 2. Fundamentalsatzes erbracht.

In der Quantenmechanik bedeuten die Zeichen $x, y, \ldots, z$ Atome, die sich zu einem Molekül zusammensetzen, $a, b, \ldots, c$ deren Valenzen. Jede Invariante der geforderten Ordnung stellt einen Spinzustand des Moleküls dar. Die durch die Monome repräsentierten „reinen Valenzzustände" veranschaulicht sich der Chemiker durch ein Valenzschema, in dem die Atome als Punkte erscheinen und jeder Klammerfaktor $[xy]$ durch einen die beiden Atome x und y verbindenden gerichteten Strich zum Ausdruck gebracht wird. $a, b, \ldots, c$ sind dann die Anzahlen der Valenzstriche, die von den einzelnen Atomen $x, y, \ldots, z$ im Valenzschema des Monoms ausgehen. Man zeichne die Punkte $x, y, \ldots, z$ auf einem Kreise auf. Die zu beweisende Regel lautet dann:

Jede Invariante J ist eine lineare Kombination solcher Monome, deren Valenzschema keine sich kreuzenden Valenzstriche enthält. Die Monome mit kreuzungslosem Valenzschema sind aber linear unabhängig von einander.

Beim Beweise des ersten Teils kann man nach dem 1. Fundamentalsatz annehmen, daß die Invariante J ein Monom ist, welches wir durch sein Valenzschema S abbilden. Es bestehe aus N Strichen zwischen den n Punkten $x, y, \ldots, z$. Wir stützen uns darauf, daß man mit Hilfe der Relation (2):

(3)



Kreuzungen auflösen kann [1]). Natürlich ist mit dieser Bemerkung nicht alles getan; denn wenn man in einem komplizierten Schema die Kreuzung zweier Valenzstriche auflöst, werden dadurch im allgemeinen andere Kreuzungen teils mit aufgelöst, teils neu entstehen. Dennoch kommt man durch ein geeignetes rekursives Arrangement zum Ziel, wie folgt.

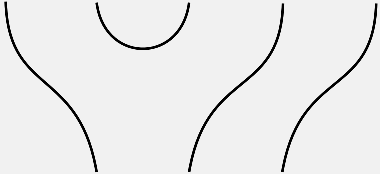
1) In der Figur wurde der Richtungssinn der Valenzstriche weggelassen.

The $\mathfrak{sl}_2$-webs

Definition(Rumar, Teller, Weyl 1932)

Fix two numbers $b, t \in \mathbb{N}$ with $b + t = 2\ell$. A $\mathfrak{sl}_2$-web w with b bottom points and t top points is an embedding (non-intersecting!) of a finite number of lines and circles in a rectangle with b fixed points at the bottom and t at the top such that the two boundary points of the lines are some of the fixed points. The set of all $\mathfrak{sl}_2$-webs w between b bottom points and t top points in denoted by $\tilde{W}_2(b, t)$.

Example($b = 3$ and $t = 5$)

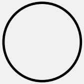


The $\mathfrak{sl}_2$-web space

Definition

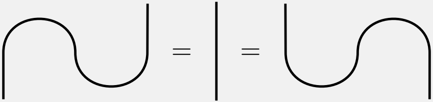
Fix two numbers $b, t \in \mathbb{N}$ with $b + t = 2\ell$. The $\mathfrak{sl}_2$-web space $W_2(b,t)$ is the free $\bar{\mathbb{Q}}(q)$-vector space generated by elements of $\tilde{W}_2(b,t)$ modulo

- The circle removal



$$= [2] = q + q^{-1}$$

- The isotopy relations



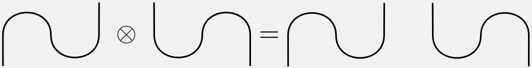
Note that $W_2(b,t)$ is a finite dimensional $\bar{\mathbb{Q}}(q)$-vector space!

The $\mathfrak{sl}_2$-web category

Definition(Kuperberg 1997)

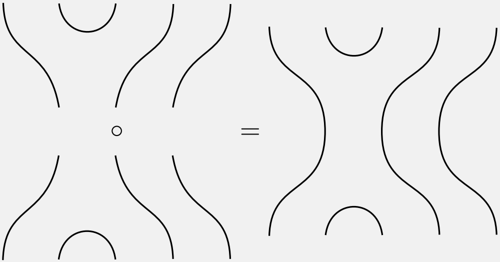
The $\mathfrak{sl}_2$-web category or web spider $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of

- The objects are the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$.
- The 1-cells $w\colon b \to t$ are the elements of $W_2(b, t)$.
- The $\bar{\mathbb{Q}}(q)$-linear composition is stacking.
- The monoidal structure $\otimes$ is given by juxtaposition, i.e. $b \otimes b' = b + b'$ and

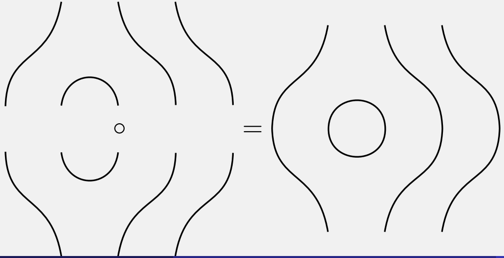


- As generators suffices the identities, shifts, cups and caps

Example



and



$= [2] \cdot \mathbf{id}(3,3)$

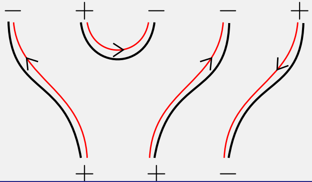
An extra information

Definition

Given a $\mathfrak{sl}_2$-web $w \in W_2(b,t)$. A $\mathfrak{sl}_2$-flow line f for w, denoted by w_f, is a choice of orientation for all lines and circles of w.
If one ignores internal circles

then such a flow line is completely determine by its boundary. There it induces a state string for the bottom $\vec{S}_b = (\pm, \dots, \pm)$ and top $\vec{S}_t = (\pm, \dots, \pm)$ with a plus for outgoing flow lines and a minus for incoming.

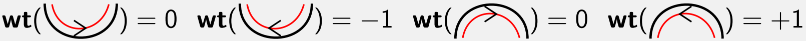
Example



The weight of a flow

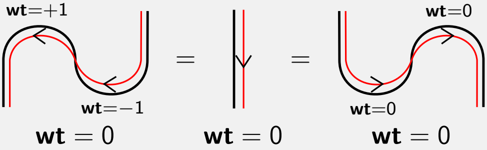
Definition

Flows on the generators of the $\mathfrak{sl}_2$-web category $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ are assigned a certain weight **wt** by the local rules



and always zero on identities and shifts. The weight of any $\mathfrak{sl}_2$-web with flow is the sum over the local weights.
Note that the weight is isotopy invariant, thus, well-defined for $\mathfrak{sl}_2$-webs without internal circles.

Example



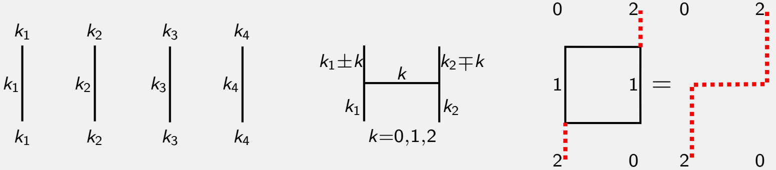
Rigidity of $\mathfrak{sl}_2$-webs

A seemingly very small point turned out to be a crucial step if we want to consider bigger n: Topology is continuous and Algebra is rigid.

Definition, second try - rigid version

The $\mathfrak{sl}_2$-web category or web spider $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of

- The objects are ordered compositions $\vec{k}$ of $2\ell \in \mathbb{N}$ with only $0,1,2$ as entries.
- The 1-cells $w\colon \vec{k} \to \vec{k}'$ are labelled ladders (we use the convention and do not draw edges labelled 0 and use a dotted line for those labelled 2) generated by juxtaposition and vertical composition of (plus relations and rest as before)

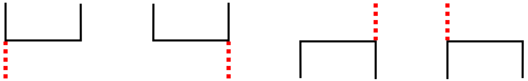


What is the upshot? "Easy" to generalize to $\mathfrak{sl}_n$: Take labels $0,1,\dots,n-1,n$ and "directly" connected to the algebra (which I explain in a second!).

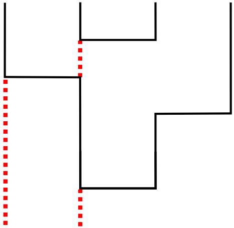
A rigid example

There is a small number of different ladders, namely the left and right shifts

and (rigid) cups and caps



These suffice to generate all $\mathfrak{sl}_2$-webs, e.g.



The quantum algebra $\mathbf{U}_q(\mathfrak{sl}_d)$

Definition

For $d \in \mathbb{N}_{>1}$ the quantum special linear algebra $\mathbf{U}_q(\mathfrak{sl}_d)$ is the associative, unital $\bar{\mathbb{Q}}(q)$-algebra generated by $K_i^{\pm 1}$ and E_i and F_i, for $i = 1, \dots, d-1$, subject to some relations (that we do not need today).

Definition(Beilinson-Lusztig-MacPherson)

For each $\vec{k} \in \mathbb{Z}^{d-1}$ adjoin an idempotent $1_{\vec{k}}$ (think: projection to the $\vec{k}$-weight space!) to $\mathbf{U}_q(\mathfrak{sl}_d)$ and add some relations, e.g.

$$1_{\vec{k}}1_{\vec{k}'} = \delta_{\vec{k},\vec{k}'}1_{\vec{k}} \text{ and } K_{\pm i}1_{\vec{k}} = q^{\pm \vec{k}_i}1_{\vec{k}} \text{ (no } K's \text{ anymore!)}.$$

The idempotented quantum special linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_d) = \bigoplus_{\vec{k},\vec{k}' \in \mathbb{Z}^{d-1}} 1_{\vec{k}}\mathbf{U}_q(\mathfrak{sl}_d)1_{\vec{k}'}.$$

The quantum algebra $\mathbf{U}_q(\mathfrak{sl}_d)$ is a Hopf algebra

It is worth noting that $\mathbf{U}_q(\mathfrak{sl}_d)$ is a Hopf algebra with coproduct Δ given by

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i,\ \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \text{ and } \Delta(K_i) = K_i \otimes K_i.$$

The antipode S and the counit ε are given by

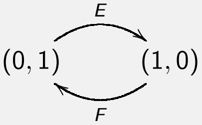
$$S(E_i) = -E_iK_i^{-1},\ S(F_i) = -K_iF_i,\ S(K_i) = K_i^{-1},\ \varepsilon(E_i) = \varepsilon(F_i) = 0,\ \varepsilon(K_i) = 1.$$

The Hopf algebra structure allows to extend actions to tensor products of representations, to duals of representations and there is a trivial representation.

Example: $\mathbf{U}_q(\mathfrak{sl}_2)$-vector representation

Consider $\bar{\mathbb{Q}}^2$ with basis $x_{-1} = (0,1), x_{+1} = (1,0)$. These are called the weights -1 and $+1$ and K acts on them by $q^{\mp 1}$. The vector representation of $\mathbf{U}_q(\mathfrak{sl}_2)$ is:

Think: $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Think: $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

The category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$

Definition

The representation category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of:

- The objects are finite tensor products of the $\mathbf{U}_q(\mathfrak{sl}_2)$-representations $\Lambda^k\bar{\mathbb{Q}}^2$. Denote them by $\vec{k} = (k_1, \dots, k_m)$ with $k_i \in \{0, 1, 2\}$.
- The 1-cells $w\colon \vec{k} \to \vec{k}'$ are $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.
- Composition of 1-cells is composition of intertwiners and $\otimes$ is the ordered tensor product.

It is worth noting that $\Lambda^0\bar{\mathbb{Q}}^2 = \bar{\mathbb{Q}}$ is the trivial $\mathbf{U}_q(\mathfrak{sl}_2)$-representation, $\Lambda^2\bar{\mathbb{Q}}^2 \cong \bar{\mathbb{Q}}$ its dual and $\Lambda^1\bar{\mathbb{Q}}^2 = \bar{\mathbb{Q}}^2$ is the (self-dual) $\mathbf{U}_q(\mathfrak{sl}_2)$-vector representation.

Example

The 1-cells of $\mathbf{Mor}(\vec{k}, \vec{k}')$ "are" (using the Hopf algebra structure!) the invariant tensors $\mathbf{Inv}_{\mathbf{U}_q(\mathfrak{sl}_2)}(\vec{k}^* \otimes \vec{k}')$ with $\vec{k}^* = (2 - k_1, \dots, 2 - k_m)$.

We have some control over the intertwiner

Given $V = \bigotimes_i \Lambda^{k_i}\bar{\mathbb{Q}}^2$ denote the tensor basis of V (recall $x_{-1} = (0,1)$ and $x_{+1} = (1,0)$, set $x_{\emptyset} = x_{\{-1,+1\}} = 1$) by $\{x_S \mid S = (S_1,\dots,S_m), S_i \subset \{-1,+1\}\}$.

Theorem(Kuperberg 1997, $n > 3$: Cautis-Kamnitzer-Morrison 2012)

Define two $\bar{\mathbb{Q}}$-linear maps called split and merge by

$$M_s^{a,b}\colon \Lambda^{a+b}\bar{\mathbb{Q}}^2 \to \Lambda^a\bar{\mathbb{Q}}^2 \otimes \Lambda^b\bar{\mathbb{Q}}^2,\ M_s^{a,b}(x_S) = \sum_{T\subset S}(-q)^{\ell(S,T)}x_T \otimes x_{S-T}$$

$$M_m^{a,b}\colon \Lambda^a\bar{\mathbb{Q}}^2\otimes\Lambda^b\bar{\mathbb{Q}}^2 \to \Lambda^{a+b}\bar{\mathbb{Q}}^2,\ M_m^{a,b}(x_S\otimes x_T) = \begin{cases}(-q)^{-\ell(T,S)}x_{S\cup T}, & S\cap T = \emptyset,\\ 0, & \text{else.}\end{cases}$$

for suitable $a, b \in \{0,1,2\}$ and $\ell(S,T) \in \{-1,0,+1\}$. These are $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiner and generate $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$.

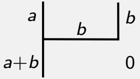
E.g.: $M_m^{1,1}(x_{-1}\otimes x_{+1}) = (-q)^0$, $M_m^{1,1}(x_{+1}\otimes x_{-1}) = (-q)^{-1}$, $M_m^{1,1}(x_{\pm 1}\otimes x_{\pm 1}) = 0$.

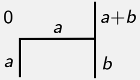
Intertwiner are pictures

Theorem(Kuperberg 1997, $n > 3$: Cautis-Kamnitzer-Morrison 2012)

The 1-categories $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ and $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ are equivalent. To be more precise, the equivalence $\Gamma\colon \mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2)) \to \mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ is given by:

- One objects: Send $\bigotimes_{\vec{k}} \Lambda^{k_i}\bar{\mathbb{Q}}^2$ to $\vec{k}$.
- One 1-cells: We only need to consider the generators split $M_s^{a,b}$ and merge $M_m^{a,b}$. Send them to

$M_s^{a,b} \mapsto$ 

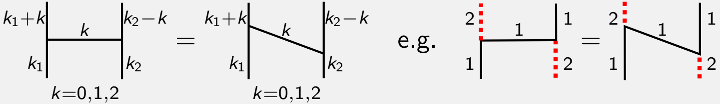
and $M_m^{a,b} \mapsto$ 

- Check that it is well-defined!

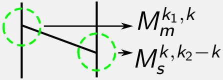
I am lying a little bit: One has to be a little more careful with objects and duals, but we ignore this for today.

Exempli gratia

What about the "left-plus-ladders"? They are a composite!

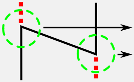


Generate them by composition of merge and split!



$$= (M_m^{k_1,k} \otimes \mathbf{id}(k_2-k, k_2-k)) \circ (\mathbf{id}(k_1,k_1) \otimes M_s^{k,k_2-k})$$

e.g.



$$= (M_m^{1,1} \otimes \mathbf{id}(1,1)) \circ (\mathbf{id}(1,1) \otimes M_s^{1,1})$$

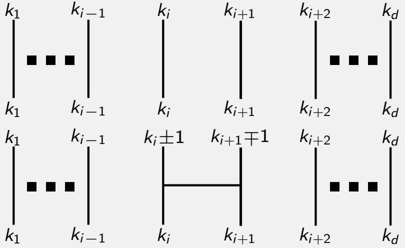
How to prove it? Quantum skew Howe duality!

Theorem

There is an $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-action on $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))^d$ (objects of length d)!

$1_{\vec{k}} \mapsto$

$E_i1_{\vec{k}},\ F_i1_{\vec{k}} \mapsto$

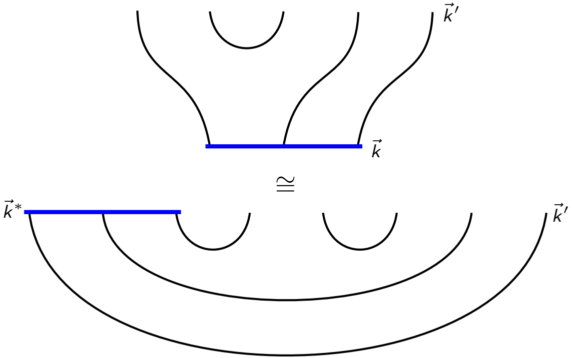


Thus, $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))^d$ is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-module and not just a $\mathbf{U}_q(\mathfrak{sl}_2)$-module.

Even better: Since, we only need "left-minus-ladders", aka F's, it can be realized as a $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-module of a certain highest weight: We can use $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-highest weight theory to prove statements about $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiner!

The invariant tensors suffice - a picture

The Hopf-structure says: $\mathbf{Mor}(\vec{k}, \vec{k}') \cong \mathbf{Inv}_{\mathbf{U}_q(\mathfrak{sl}_2)}(\vec{k}^* \otimes \vec{k}')$. The picture says:



Remaining question: How to identify the invariant tensors?

This is not trivial...

First question: What do we mean by "identify" the invariant tensors?

$$\text{Recall: } \mathbf{Inv}_{\mathbf{U}_q(\mathfrak{sl}_2)}(\Lambda^{k_1}\bar{\mathbb{Q}}^2 \otimes \cdots \otimes \Lambda^{k_d}\bar{\mathbb{Q}}^2) \subset \Lambda^{k_1}\bar{\mathbb{Q}}^2 \otimes \cdots \otimes \Lambda^{k_d}\bar{\mathbb{Q}}^2,$$

and $\Lambda^{k_1}\bar{\mathbb{Q}}^2 \otimes \cdots \otimes \Lambda^{k_d}\bar{\mathbb{Q}}^2$ has a easy to write down, but horrible to work with basis: The elementary tensors $x_{\vec{S}}$!

Thus, "identify" $v \in \mathbf{Inv}_{\mathbf{U}_q(\mathfrak{sl}_2)}(\Lambda^{k_1}\bar{\mathbb{Q}}^2 \otimes \cdots \otimes \Lambda^{k_d}\bar{\mathbb{Q}}^2)$ is writing v in terms of $x_{\vec{S}}$.
Second question: How to do it? Recall that the action on tensors is given by

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i,\ \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \text{ and } \Delta(K_i) = K_i \otimes K_i.$$

Exempli gratia

Example: $\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2$

Recall that $\bar{\mathbb{Q}}^2$ has basis $x_{+1} = (1,0)$ and $x_{-1} = (0,1)$. Thus, $\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2$ has basis

$\{x_{+1+1} = x_{+1} \otimes x_{+1}, x_{+1-1} = x_{+1} \otimes x_{-1}, x_{-1+1} = x_{-1} \otimes x_{+1}, x_{-1-1} = x_{-1} \otimes x_{-1}\}$.

Test calculation:

$$\begin{aligned} F \cdot x_{+1-1} &= F \cdot x_{+1} \otimes x_{-1} + K^{-1} \cdot x_{+1} \otimes F \cdot x_{-1} \\ &= x_{-1} \otimes x_{-1} \\ F \cdot x_{-1+1} &= F \cdot x_{-1} \otimes x_{+1} + K^{-1} \cdot x_{-1} \otimes F \cdot x_{+1} \\ &= q^{+1} x_{-1} \otimes x_{-1} \end{aligned}$$

Claim: $x_{+1,-1} - q^{-1} x_{-1+1}$ is invariant and spans $\mathbf{Inv}_{\mathbf{U}_q(\mathfrak{sl}_2)}(\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2)$.

How to do this in general?

The diagrammatic calculus helps

Theorem(Khovanov-Kuperberg 1997)

The decomposition of a $\mathfrak{sl}_2$-web $w \in \mathbf{Mor}(\emptyset, \vec{k})$ in terms of the elementary tensors $x_{\vec{S}}$ is encoded by the flow lines f on w in the following way:

- Each flow f induces a state string $\vec{S}_f = (\pm, \dots, \pm)$ at the boundary and has a weight $\mathbf{wt}(w_f)$.
- Then the coefficient for $x_{\vec{S}_f}$ is $(-q)^{\mathbf{wt}(w_f)}$.
- Thus, $w = \sum_f (-q)^{\mathbf{wt}(w_f)} x_{\vec{S}_f}$.
- (Only $n = 2$!) A basis Arc of $\mathbf{Mor}(\emptyset, \vec{k})$ is given by all arc diagrams.

Example: $\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2$ again

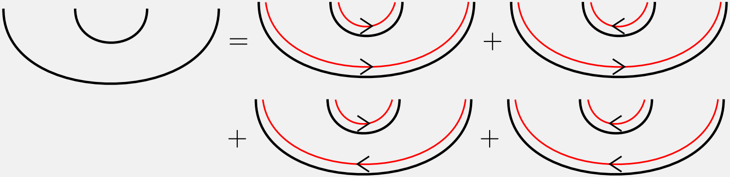
In this case there is exactly one arc u and it has the two flows

$\mathbf{wt}($  $) = 0, \vec{S} = (+1, -1)$ and $\mathbf{wt}($  $) = -1, \vec{S} = (-1, +1)$

Conclusion: $u = x_{+1} \otimes x_{-1} - q^{-1} x_{-1} \otimes x_{+1}$.

The negative exponent property

Example



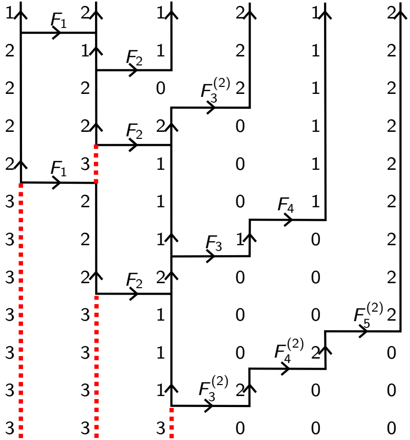
$$= x_{+1} \otimes x_{+1} \otimes x_{-1} \otimes x_{-1} - q^{-1} x_{+1} \otimes x_{-1} \otimes x_{+1} \otimes x_{-1}$$
$$- q^{-1} x_{-1} \otimes x_{+1} \otimes x_{-1} \otimes x_{+1} + q^{-2} x_{-1} \otimes x_{-1} \otimes x_{+1} \otimes x_{+1}$$

Observation: One leading term plus a rest with coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$! This is called the negative exponent property. In fact, the arc basis is the dual canonical basis in the sense of Lusztig.

$n > 2$? Use q-skew Howe!

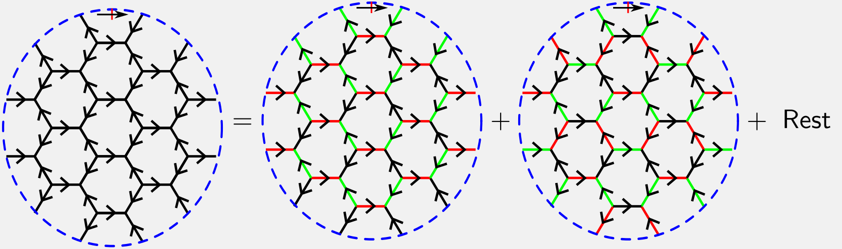
Roughly:

- Express a $\mathfrak{sl}_n$-web as a string of F's acting on a highest weight vector v_{n^ℓ}.
- The action of the F's is given by the splits and merges. Read of the resulting vector inductively.
- There is also a purely combinatorial way to do this!



Dual canonical $\mathfrak{sl}_n$-webs? Quantum skew Howe duality!

Counterexample



$$= x^2_{+1+1} \otimes x^1_{+1+1} \otimes x^2_{00} \otimes x^1_{00} \otimes x^2_{-1-1} \otimes x^1_{-1-1}$$
$$+ x^2_{+1-1} \otimes x^1_{+1-1} \otimes x^2_{+1-1} \otimes x^1_{+1-1} \otimes x^2_{+1-1} \otimes x^1_{+1-1} + \text{ Rest}$$

Here Rest has coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$.

Note that "most" $n > 2$-webs do not have this property and this makes live very complicated! But using q-skew Howe duality one can obtain an iff-condition for a web to be dual canonical plus an algorithm to compute the dual canonical basis.

The famous Jones polynomial

Let L_D be a diagram of an oriented link. Set $[2] = q + q^{-1}$ and

$$n_+ = \text{number of crossings } \nearrow\!\!\!\!\nwarrow \qquad n_- = \text{number of crossings } \nwarrow\!\!\!\!\nearrow$$

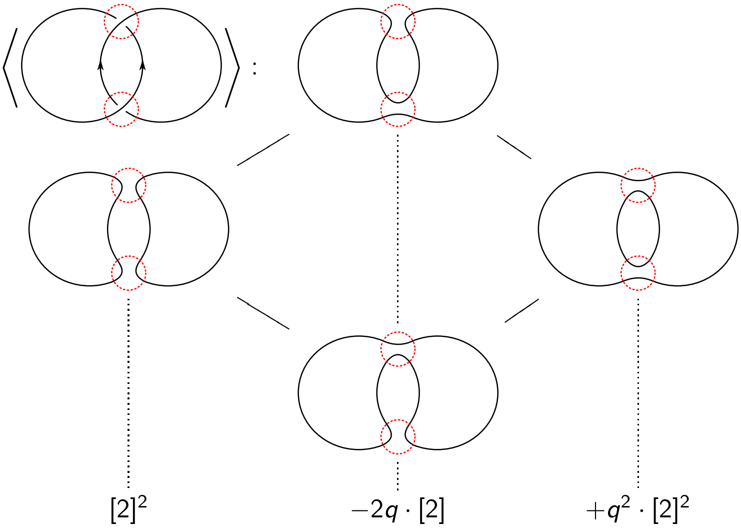
Definition/Theorem(Jones 1984, Kauffman 1987)

The bracket polynomial of the diagram L_D (without orientations) is a polynomial $\langle L_D \rangle \in \mathbb{Z}[q, q^{-1}]$ given by the following rules.

- $\langle \emptyset \rangle = 1$ (normalization).
- $\langle \times \rangle = \langle \rangle \; \langle \rangle - q \langle \asymp \rangle$ (recursion step 1).
- $\langle \bigcirc \amalg L_D \rangle = [2] \cdot \langle L_D \rangle$ (recursion step 2).
- $[2] J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$ (Re-normalization).

The polynomial $J(\cdot) \in \mathbb{Z}[q, q^{-1}]$ is an invariant of oriented links.

Exempli gratia



Thus, $J(\textbf{Hopf}) = q^5 + q$, i.e the Hopf link is not trivial!

“The Jones revolution”

Definition/Theorem(HOMFLY 1985, PT 1987)

Define a polynomial $P_n(L_D) \in \mathbb{Z}[q, q^{-1}]$ uniquely determined by the property $P_n(\bigcirc) = 1$ and the so-called $\mathfrak{sl}_n$ skein relations

$$q^{2n} \cdot P_n(\nearrow\!\!\!\!\nwarrow) - q^{-2n} \cdot P_n(\nwarrow\!\!\!\!\nearrow) = (q + q^{-1}) \cdot P_n(\uparrow\,\uparrow).$$

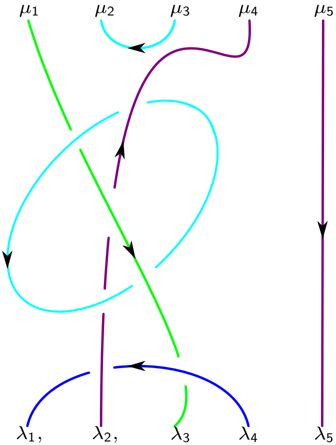
The $\mathfrak{sl}_n$-HOMFLY-PT polynomial is a link invariant and $P_2(L_D) = J(L_D)$.

Shortly after Jones several authors independently found new knot polynomials. One example is the HOMFLY-PT polynomial. Moreover, researches discovered connections to different parts of mathematics and physics. Before the “Jones revolution” there was a lack of knot polynomials and after there where too many. The questions shifted to:

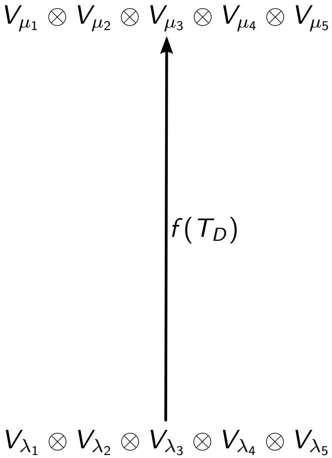
“Why do they exist? How can we order them?”

A tangle is an intertwiner

Let $\mathfrak{g}$ be any classical Lie algebra. Denote by λ_i, μ_j the $\mathbf{U}_q(\mathfrak{g})$-representation of highest weight V_{λ_i}, V_{μ_j}. Let T_D be a diagram of a, λ_i, μ_j-colored, oriented tangle.



$\rightsquigarrow$



Representation theory does the trick!

Definition(Reshetikhin-Turaev 1990)

Given the set-up from before we define a certain $\mathbf{U}_q(\mathfrak{g})$-intertwiner

$$f(T_D)\colon V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k} \to V_{\lambda_{k+1}} \otimes \cdots \otimes V_{\lambda_l}.$$

Theorem(Reshetikhin-Turaev 1990)

The $\mathbf{U}_q(\mathfrak{g})$-intertwiner $f(T_D)$ is an invariant of T_D.

Corollary(Reshetikhin-Turaev 1990)

In the case of colored, oriented links L_D we have

$$f(L_D)\colon \bar{\mathbb{Q}}(q) \to \bar{\mathbb{Q}}(q),\ 1 \mapsto P_{\mathbf{RT}}(L_D) \in \mathbb{Z}[q, q^{-1}],$$

that is each configuration as above gives a polynomial invariant of oriented links!

This is powerful!

Example

We have the following list of examples!

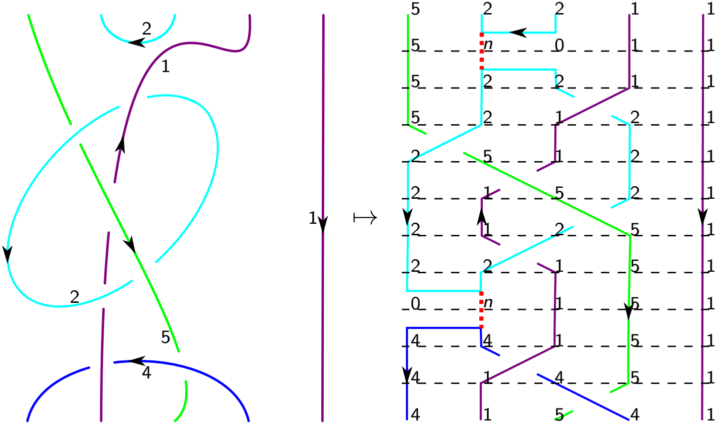
- Let $\mathfrak{g} = \mathfrak{sl}_2$. If we restrict to the $\mathbf{U}_q(\mathfrak{sl}_2)$-vector representation $\bar{\mathbb{Q}}^2$, then the Reshetikhin-Turaev polynomial $P_{\mathbf{RT}}(\cdot)$ is the Jones or $\mathfrak{sl}_2$-polynomial.
- Let $\mathfrak{g} = \mathfrak{sl}_2$. If we allow any kind of coloring with $\mathbf{U}_q(\mathfrak{sl}_2)$-representations, then $P_{\mathbf{RT}}(\cdot)$ is the so-called colored Jones polynomial.
- Let $\mathfrak{g} = \mathfrak{sl}_n$. If we restrict to the $\mathbf{U}_q(\mathfrak{sl}_n)$-vector representation $\bar{\mathbb{Q}}^n$, then the Reshetikhin-Turaev polynomial $P_{\mathbf{RT}}(\cdot)$ is the $\mathfrak{sl}_n$-polynomial.
- But the Reshetikhin-Turaev polynomial is much more generalize than all of them and "explains" them using one concept.

Moral: A lot of link polynomials are special instances of symmetries of the quantum groups $\mathbf{U}_q(\mathfrak{g})$!

Question: Can we do this more explicit for $\mathfrak{g} = \mathfrak{sl}_n$?

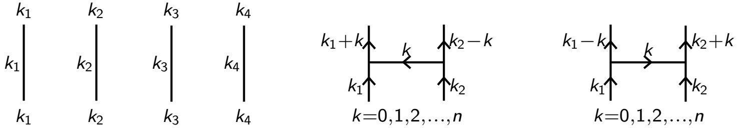
"Straightening" again

Consider a diagram of an oriented tangle. Its components can be colored with colors $k \in \{0, \ldots, n\}$. These colors correspond to the fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representations $\Lambda^k \bar{\mathbb{Q}}^n$. Straightening it into a Morse position.

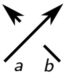


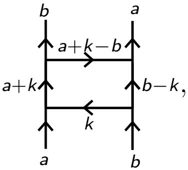
Tangles to $\mathfrak{sl}_n$-webs

We can define as before the category $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n))$ consisting of 1-cells as



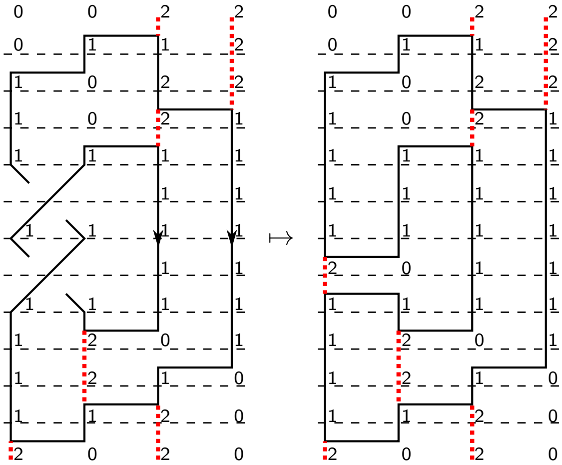
Let $b \leq a$. Define an $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiner $\Lambda^a\bar{\mathbb{Q}}^n \otimes \Lambda^b\bar{\mathbb{Q}}^n \to \Lambda^b\bar{\mathbb{Q}}^n \otimes \Lambda^a\bar{\mathbb{Q}}^n$ as follows.



$$= \sum_{k=0}^{b} (-1)^{k+(a+1)b} q^{-b+k}$$


"Morally" (up to some signs, shifts, re-orientations) the same for $a < b$ and .

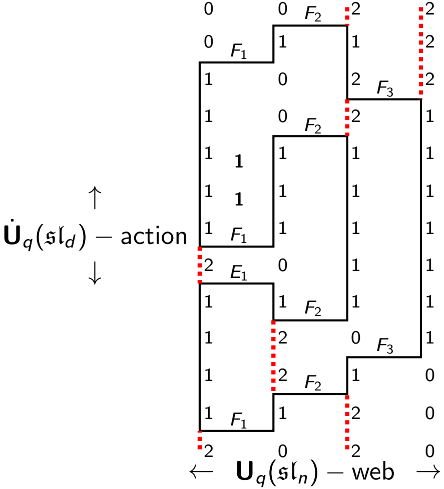
Exempli gratia: Hopf link for $\mathfrak{sl}_2$



$f_{10}(\mathbf{Hopf})\colon \Lambda^2\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}} \otimes \Lambda^2\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}} \to \bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}} \otimes \Lambda^2\bar{\mathbb{Q}}^2 \otimes \Lambda^2\bar{\mathbb{Q}}^2$ is an intertwiner in $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n))$. In the end we get the same polynomial as before (up to a shift).

Conclusion: Works fine for $n = 2$. What about $n > 2$?

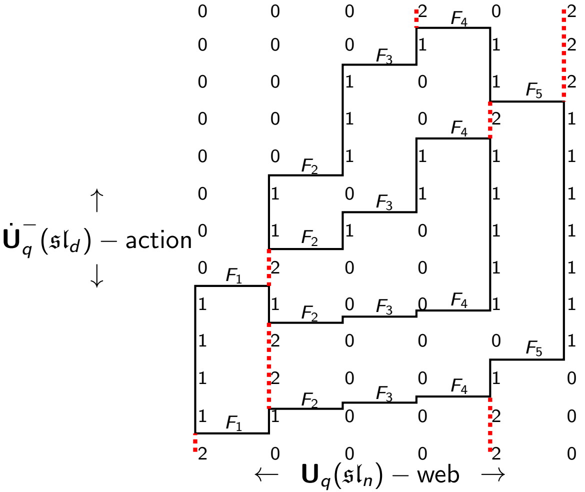
Quantum skew Howe duality helps



Recall that we have an $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-action on $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n))^d$. In the example above

$$f_{10}(\mathbf{Hopf}) = F_2F_1F_3F_2F_1E_1F_2F_3F_2F_1F_2^{(2)}v_{2200}.$$

The lower part $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_d)$ suffices!



A crucial observation: We need only F's!

$$f_{10}(\mathbf{Hopf}) = F_4^{(2)}F_4F_3F_5F_4F_2F_3F_2F_1F_4F_3F_2F_5F_4F_3F_2F_1F_4^{(2)}F_3^{(2)}F_2^{(2)}v_{220000}.$$

The $\mathfrak{sl}_n$-polynomials using $\mathfrak{sl}_d$-symmetries

Let us summarize the connection between (colored) $\mathfrak{sl}_n$-polynomials and the $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-$\mathbf{U}_q(\mathfrak{sl}_n)$-skew Howe duality.

- Reshetikhin-Turaev: The $\mathfrak{sl}_n$-polynomials $P_n(\cdot)$ are $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiner.
- $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiner are vectors in hom's between $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-weight spaces.
- Only F's: The space of invariant $\mathbf{U}_q(\mathfrak{sl}_n)$-tensors is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-representation of some highest weight v_h and $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_d)$ suffices.
- Conclusion: The (colored) $\mathfrak{sl}_n$-polynomials $P_n(\cdot)$ are instances of highest $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-weight representation theory!
- If L_D is a link diagram, then $P_n(L_D)$ is obtained by jumping via F's from a highest $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-weight v_h to a lowest $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-weight v_l!

There is still much to do...

Thanks for your attention!