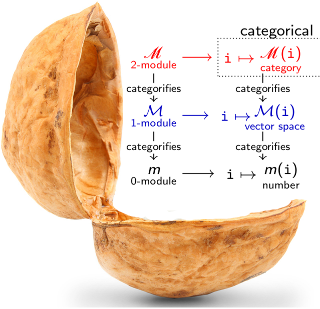
What is...2-representation theory?

Or: Why do I care?

Daniel Tubbenhauer



categorical representation

$$\begin{tikzcd}
\underset{\text{2-module}}{\mathscr{M}} \arrow[r] \arrow[d, "\text{categorifies}"] & \mathrm{i} \mapsto \underset{\text{category}}{\mathscr{M}(\mathrm{i})} \arrow[d, "\text{categorifies}"] & \mathrm{F} \mapsto \underset{\text{functor}}{\mathscr{M}(\mathrm{F})} \arrow[d, "\text{categorifies}"] & \alpha \mapsto \underset{\text{nat. trafo}}{\mathscr{M}(\alpha)} \\
\underset{\text{1-module}}{\mathcal{M}} \arrow[r] \arrow[d, "\text{categorifies}"] & \mathrm{i} \mapsto \underset{\text{vector space}}{\mathcal{M}(\mathrm{i})} \arrow[d, "\text{categorifies}"] & \mathrm{F} \mapsto \underset{\text{linear map}}{\mathcal{M}(\mathrm{F})} & \\
\underset{\text{0-module}}{m} \arrow[r] & \mathrm{i} \mapsto \underset{\text{number}}{m(\mathrm{i})} & &
\end{tikzcd}$$

Representation theory is group theory in vector spaces

Let C be a group, an algebra *etc.*

Frobenius $\sim$1895++, Burnside $\sim$1900++, Noether $\sim$1928++.
Representation theory is the study of actions

$$\mathcal{M}\colon \mathrm{C} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}),$$

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

Basic question: Try to develop a reasonable theory of such actions.

Examples.

- **Weyl $\sim$1923++.** The representation theory of (semi)simple Lie groups.
- **Noether $\sim$1928++.** The representation theory of finite-dimensional algebras.

2-representation theory is group theory in categories

Let C be a reasonable 2-category.

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. 2-representation theory is the study of 2-actions of 2-categories:

$$\mathbf{M}\colon \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathbf{V}),$$

with **V** being some finitary category. (Called 2-modules or 2-representations.)

Basic question: Try to develop a reasonable theory of such 2-actions.

Examples.

- **Chuang–Rouquier & Khovanov–Lauda style.** The 2-representation theory of (semi)simple Lie groups. Another time.
- **Abelian ∼2000++ or additive ∼2010++.** The 2-representation theory of finite-dimensional algebras. Today.

2-representation theory is group theory in categories

Let C be a reasonable 2-category.

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. 2-representation theory is the study of 2-actions of 2-categories:

$$\mathbf{M}\colon \mathscr{C} \to \mathcal{E}nd(\mathbf{V})$$

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Empirical fact.

Most of the fun happens already for monoidal categories (one-object 2-categories);

I will stick to this case for the rest of the talk,

but what I am going to explain works for 2-categories.

Examples.

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- **Abelian ∼2000++ or additive ∼2010++.** The 2-representation theory of finite-dimensional algebras. Today.

Abelian vs. additive a.k.a. “What are the elements?”.

Finite tensor categories—the abelian world.

- Elements are simple objects. Finite means finitely many of these.
- What acts are finite multitensor categories $\mathscr{C}$, *i.e.* finite abelian, $\mathbb{K}$-linear, rigid (without duality all hope is lost) monoidal categories, with $\otimes\colon \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ being bilinear.
- We act on finite abelian, $\mathbb{K}$-linear categories $\mathbf{V}$, with the 2-action $\otimes\colon \mathscr{C}\times\mathbf{V}\to\mathbf{V}$ being bilinear and biexact.
- The abelian Grothendieck groups are finite-dimensional algebras or finite-dimensional modules of such, respectively.

Examples.

- Finite-dimensional vector spaces, or any fusion category (fusion=finite tensor+semisimple).
- Modules of finite groups, or more generally, of finite-dimensional Hopf algebras.
- We see examples of 2-modules momentarily.

Abelian vs. additive a.k.a. “What are the elements?”.

Finite additive means
additive
finitely many indecomposables
finite-dimensional hom-spaces
Krull–Schmidt.

Fiat 2-categories—the additive world.

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- We act on finite additive, $\mathbb{K}$-linear categories $\mathbf{V}$, with the 2-action $\otimes\colon \mathscr{C}\times\mathbf{V}\to\mathbf{V}$ being bilinear.
- The additive Grothendieck groups are finite-dimensional algebras or finite-dimensional modules of such, respectively.

Examples.

- Finite-dimensional vector spaces, or any fusion category (fusion=fiat+semisimple).
- Modules of finite groups of finite representation type, or more generally, of finite-dimensional Hopf algebras of finite representation type.
- Projective/injective modules of finite groups of finite representation type, or more generally, of finite-dimensional Hopf algebras.

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  finite-dimensional modules of such, respectively.

Abelian and additive run in parallel,

but additive is harder, *e.g.* no version of Schur’s 2-lemma.

▸ Another point why additive is harder.

Examples.

- Finite-dimensional vector spaces, or any fusion category (fusion=fiat+semisimple).
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- Projective/injective modules of finite groups of finite representation type, or more generally, of finite-dimensional Hopf algebras.

Take your favorite theorem and categorify it.

Some facts run in parallel, e.g.

the regular module $\mathcal{M}\colon \mathrm{C} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V})$, $a \mapsto a \cdot \_$	the regular 2-module $\mathbf{M}\colon \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathbf{V})$, $\mathtt{M} \mapsto \mathtt{M} \otimes \_$
simples (no non-trivial $\mathscr{C}$-stable subspace) and Jordan–Hölder	2-simples (no non-trivial $\mathscr{C}$-stable ideal) and 2-Jordan–Hölder
double centralizer theorem, *i.e.* $\mathrm{C} \cong \mathcal{E}\mathrm{nd}_{\mathcal{E}\mathrm{nd}_{\mathrm{C}}(\mathtt{V})}(\mathtt{V})$ for $\mathtt{V}$ being faithful.	2-double centralizer theorem, *i.e.* $\mathscr{C} \cong \mathscr{E}\mathrm{nd}_{\mathscr{E}\mathrm{nd}_{\mathscr{C}}(\mathbf{V})}(\mathbf{V})$ for $\mathbf{V}$ being 2-faithful. (Theorem 2020)

Some do not, e.g.

Schur's lemma, *i.e.* hom-spaces between simples are trivial	hom-spaces between 2-simples can be arbitrary complicated
there are finitely many simples	there can be ∞ many 2-simples

Example ($\mathscr{R}\mathrm{ep}(G)$, ground field $\mathbb{C}$).

- Let $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$, for G being a finite group.
- $\mathscr{C}$ is fusion: For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

$$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial module $\mathbb{1}$.
- The regular 2-module $\mathbf{M} \colon \mathscr{C} \to \mathscr{E}\mathrm{nd}(\mathscr{C})$:

$$\begin{tikzcd}
\mathtt{M} \arrow[r] \arrow[d, "f"'] & \mathtt{M} \otimes \_ \arrow[d, "f \otimes \_"] \\
\mathtt{N} \arrow[r] & \mathtt{N} \otimes \_
\end{tikzcd} .$$

- The decategorification is a $\mathbb{N}$-module, the regular module.

Example ($\mathscr{R}\mathrm{ep}(G)$, ground field $\mathbb{C}$).

- Let $K \subset G$ be a subgroup.
- **Rep**(K) is a 2-module of $\mathscr{R}\mathrm{ep}(G)$, with 2-action

$$\mathcal{R}es_K^G \otimes \_ \colon \mathscr{R}\mathrm{ep}(G) \to \mathscr{E}\mathrm{nd}\big(\mathbf{Rep}(K)\big),$$

$$\begin{tikzcd}
\mathtt{M} \arrow[r] \arrow[d, "f"'] & \mathcal{R}es_K^G(\mathtt{M}) \otimes \_ \arrow[d, "\mathcal{R}es_K^G(f)\otimes\_"] \\
\mathtt{N} \arrow[r] & \mathcal{R}es_K^G(\mathtt{N}) \otimes \_
\end{tikzcd} .$$

  which is indeed a 2-action because $\mathcal{R}es_K^G$ is a $\otimes$-functor.
- The decategorifications are $\mathbb{N}$-modules.

Example ($\mathscr{R}\mathrm{ep}(G)$, ground field $\mathbb{C}$).

- Let $\psi \in H^2(K, \mathbb{C}^*)$. Let $\mathbf{V}(K, \psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.* a vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

$$\rho(g)\rho(h) = \psi(g, h)\rho(gh), \text{ for all } g, h \in K.$$

- Note that $\mathbf{V}(K, 1) = \mathbf{Rep}(K)$ and

$$\otimes\colon \mathbf{V}(K, \phi) \boxtimes \mathbf{V}(K, \psi) \to \mathbf{V}(K, \phi\psi).$$

- $\mathbf{V}(K, \psi)$ is also a 2-module of $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$:

$$\mathscr{R}\mathrm{ep}(G) \ \boxtimes\ \mathbf{V}(K, \psi) \xrightarrow{\mathcal{R}es_K^G \boxtimes \mathrm{Id}} \mathbf{Rep}(K) \ \boxtimes\ \mathbf{V}(K, \psi) \xrightarrow{\otimes} \mathbf{V}(K, \psi).$$

- The decategorifications are $\mathbb{N}$-modules.

Example ($\mathscr{R}\mathrm{ep}(G)$, ground field $\mathbb{C}$).

Theorem (folklore?).

Completeness. All 2-simples of $\mathscr{R}\mathrm{ep}(G)$ are of the form $\mathbf{V}(K,\psi)$.

Non-redundancy. We have $\mathbf{V}(K,\psi)\cong\mathbf{V}(K',\psi')$

$\Leftrightarrow$

the subgroups are conjugate or $\psi'=\psi^{g}$, where $\psi^{g}(k,l)=\psi(gkg^{-1},glg^{-1})$.

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Note that $\mathscr{R}\mathrm{ep}(G)$ has only finitely many 2-simples. ▸ Example

This is no coincidence.

- $\mathbf{V}(K,\psi)$ is also a 2-module of $\mathscr{C}=\mathscr{R}\mathrm{ep}(G)$:

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This is no coincidence.

- $\mathbf{V}(K,\psi)$ is also a 2-module of $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$:

Theorem (Etingof–Nikshych–Ostrik ∼2004).

If $\mathscr{C}$ is fusion (fiat and semisimple),
then it has only finitely many 2-simples. This is false if one drops semisimplicity.

Theorem (2020).

The non-semisimple, non-abelian Hecke category has only finitely many 2-simples.

There is still much to do...

Why 2-representation theory?

Or: Representation theory of the 21th century!?

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Thanks for your attention!

Example. $\mathscr{R}\mathrm{ep}(\mathbb{Z}/5\mathbb{Z})$ in characteristic 5.

- ▷ Indecomposables correspond to Jordan blocks of $\overline{\mathbb{F}}_5[X]/(X^5) \cong \overline{\mathbb{F}}_5(\mathbb{Z}/5\mathbb{Z})$:

$$\mathrm{Z}_1 \leftrightsquigarrow X \mapsto \begin{pmatrix} 0 \end{pmatrix}, \quad \mathrm{Z}_2 \leftrightsquigarrow X \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{Z}_3 \leftrightsquigarrow X \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{Z}_4 \leftrightsquigarrow X \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{Z}_5 \leftrightsquigarrow X \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

  $\Rightarrow \mathscr{R}\mathrm{ep}(\mathbb{Z}/5\mathbb{Z})$ has five elements as an additive category.

- ▷ Only Z_1 is simple $\Rightarrow \mathscr{R}\mathrm{ep}(\mathbb{Z}/5\mathbb{Z})$ has only one element as an abelian category.
- ▷ Only Z_5 is projective $\Rightarrow \mathscr{P}\mathrm{roj}(\mathbb{Z}/5\mathbb{Z}) = \mathscr{I}\mathrm{nj}(\mathbb{Z}/5\mathbb{Z})$ has one element as an additive category, and $\mathscr{P}\mathrm{roj}(\mathbb{Z}/5\mathbb{Z})$ not abelian.

In characteristic $\neq 5$ we have $\mathscr{R}\mathrm{ep}(\mathbb{Z}/5\mathbb{Z}) = \mathscr{P}\mathrm{roj}(\mathbb{Z}/5\mathbb{Z}) = \mathscr{I}\mathrm{nj}(\mathbb{Z}/5\mathbb{Z})$ and there is no difference between tensor (abelian) and fiat (additive).

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For example, for $\mathscr{R}\mathrm{ep}(S_5)$ we have:

$\mathscr{R}\mathrm{ep}(S_5)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5
$\#$	1	2	1	1	2	1	2	1	1	1	1	1	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	$4, 1$	5	3	6	$5, 2$	$4, 2$	$4, 3$	$6, 3$	5	$5, 3$	$5, 4$	$7, 5$

This is completely different from their classical representation theory of S_5.

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