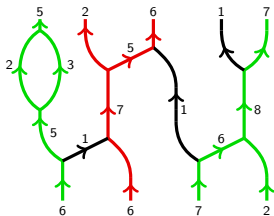


# $U_q(\mathfrak{sl}_N)$ diagram categories via $q$ -Howe duality

Or: from dualities to diagrams

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Joint work with David Rose, Pedro Vaz and Paul Wedrich

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- 1 Exterior  $\mathfrak{sl}_N$ -web categories
  - Graphical calculus via Temperley-Lieb diagrams
  - Its cousins: the  $N$ -webs<sub>g</sub>
  - Proof? Skew quantum Howe duality!
- 2 Symmetric  $\mathfrak{sl}_2$ -web categories
  - More cousins: the 2-webs<sub>r</sub>
  - Proof? Symmetric quantum Howe duality!
- 3 Exterior-symmetric  $\mathfrak{sl}_N$ -web categories
  - Even more cousins: the  $N$ -webs<sub>gr</sub>
  - Proof? Super quantum Howe duality!
  - Green-red symmetry and the Hecke algebroid

## Definition (Rumer-Teller-Weyl 1932)

The  $2\text{-web}_g$  space  $\text{Hom}_{2\text{-Web}_g}(b, t)$  is the free  $\mathbb{C}_q = \mathbb{C}(q)$ -vector space generated by non-intersecting arc diagrams with  $b, t$  bottom/top boundary points modulo:

- The *circle removal*:

$$1 \bigcirc = -q - q^{-1} = -[2]$$

- The *isotopy relations*:

The diagram shows three green arcs on a white background, connected by equals signs. The first arc starts at a bottom point labeled '1', goes up, loops to the left, crosses itself, loops to the right, and ends at a top point labeled '1'. The second arc is a simple vertical line from a bottom point labeled '1' to a top point labeled '1'. The third arc starts at a top point labeled '1', goes down, loops to the right, crosses itself, loops to the left, and ends at a bottom point labeled '1'.

# The $2\text{-web}_g$ category

## Definition (Kuperberg 1995)

The  $2\text{-web}_g$  category  $2\text{-Web}_g$  is the (braided) monoidal,  $\mathbb{C}_q$ -linear category with:

- Objects are vectors  $\vec{k} = (1, \dots, 1)$  and morphisms are  $\text{Hom}_{2\text{-Web}_g}(\vec{k}, \vec{l})$ .
- Composition  $\circ$ :

$$\begin{array}{c} \text{cap} \\ \text{---} \\ \text{cup} \end{array} \circ \begin{array}{c} \text{cup} \\ \text{---} \\ \text{cap} \end{array} = \text{circle} \quad , \quad \begin{array}{c} \text{cup} \\ \text{---} \\ \text{cap} \end{array} \circ \begin{array}{c} \text{cap} \\ \text{---} \\ \text{cup} \end{array} = \text{two lines}$$

- Tensoring  $\otimes$ :

$$\begin{array}{c} \text{cup} \\ \text{---} \\ \text{cap} \end{array} \otimes \begin{array}{c} \text{cap} \\ \text{---} \\ \text{cup} \end{array} = \begin{array}{c} \text{cup} \\ \text{---} \\ \text{cap} \end{array}$$

# Diagrams for intertwiners

Observe that there are (up to scalars) unique  $\mathbf{U}_q(\mathfrak{sl}_2)$ -intertwiners

$$\text{cap}: \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \rightarrow \mathbb{C}_q \quad \text{and} \quad \text{cup}: \mathbb{C}_q \rightarrow \mathbb{C}_q^2 \otimes \mathbb{C}_q^2,$$

projecting  $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$  onto  $\mathbb{C}_q$  respectively embedding  $\mathbb{C}_q$  into  $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$ .

Let  $\mathfrak{sl}_2\text{-Mod}_e$  be the (braided) monoidal,  $\mathbb{C}_q$ -linear category whose objects are tensor generated by  $\mathbb{C}_q^2$ . Define a functor  $\Gamma: 2\text{-Web}_g \rightarrow \mathfrak{sl}_2\text{-Mod}_e$ :

- On objects:  $\vec{k} = (1, \dots, 1)$  is sent to  $(\mathbb{C}_q^2)^{\otimes k} = \mathbb{C}_q^2 \otimes \dots \otimes \mathbb{C}_q^2$ .
- On morphisms:

$$\begin{array}{c} \text{cap} \\ \text{cup} \end{array} \mapsto \text{cap} \quad , \quad \begin{array}{c} \text{cup} \\ \text{cap} \end{array} \mapsto \text{cup}$$

## Theorem(Folklore)

$\Gamma: 2\text{-Web}_g^{\oplus} \rightarrow \mathfrak{sl}_2\text{-Mod}_e$  is an equivalence of (braided) monoidal categories.

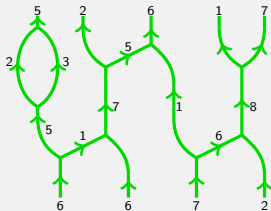
# The main step beyond $\mathfrak{sl}_2$ : trivalent vertices

An  $N$ -web<sub>g</sub> is an oriented, labeled, trivalent graph locally made of

$$m_{k,l}^{k+l} = \begin{array}{c} k+l \\ \uparrow \\ \begin{array}{cc} \nearrow & \nwarrow \\ k & l \end{array} \end{array}, \quad s_{k+l}^{k,l} = \begin{array}{c} k \quad l \\ \nearrow \quad \nwarrow \\ \uparrow \\ k+l \end{array} \quad k, l, k+l \in \mathbb{N}$$

(and no pivotal things today).

## Example



# Let us try the same for $\mathfrak{sl}_N$ : the $N$ -web $_g$ space

Define the (braided) monoidal,  $\mathbb{C}_q$ -linear category  $N\text{-Web}_g$  by using:

## Definition (Cautis-Kamnitzer-Morrison 2012)

The  $N$ -web $_g$  space  $\text{Hom}_{N\text{-Web}_g}(\vec{k}, \vec{l})$  is the free  $\mathbb{C}_q$ -vector space generated by  $N$ -webs $_g$  with  $\vec{k}$  and  $\vec{l}$  at the bottom and top modulo isotopies and:

- “gl $_m$  ladder” relations like

$$\begin{array}{c} k \\ \uparrow \\ \text{---} \\ \uparrow \\ k-1 \end{array} \begin{array}{c} l \\ \uparrow \\ \text{---} \\ \uparrow \\ l+1 \end{array} - \begin{array}{c} k \\ \uparrow \\ \text{---} \\ \uparrow \\ k+1 \end{array} \begin{array}{c} l \\ \uparrow \\ \text{---} \\ \uparrow \\ l-1 \end{array} = [k - l] \begin{array}{c} k \\ | \\ k \end{array} \begin{array}{c} l \\ | \\ l \end{array}$$

- The exterior relations:

$$\begin{array}{c} | \\ \uparrow \\ k \end{array} = 0, \quad \text{if } k > N.$$

## Diagrams for intertwiners - Part 2

Observe that there are (up to scalars) unique  $\mathbf{U}_q(\mathfrak{sl}_N)$ -intertwiners

$$m_{k,l}^{k+l}: \Lambda_q^k \mathbb{C}_q^N \otimes \Lambda_q^l \mathbb{C}_q^N \rightarrow \Lambda_q^{k+l} \mathbb{C}_q^N \quad \text{and} \quad s_{k+l}^{k,l}: \Lambda_q^{k+l} \mathbb{C}_q^N \rightarrow \Lambda_q^k \mathbb{C}_q^N \otimes \Lambda_q^l \mathbb{C}_q^N$$

given by projection and inclusion again.

Let  $\mathfrak{sl}_N\text{-Mod}_e$  be the (braided) monoidal,  $\mathbb{C}_q$ -linear category whose objects are tensor generated by  $\Lambda_q^k \mathbb{C}_q^N$ . Define a functor  $\Gamma: N\text{-Web}_g \rightarrow \mathfrak{sl}_N\text{-Mod}_e$ :

- On objects:  $\vec{k} = (k_1, \dots, k_m)$  is sent to  $\Lambda_q^{k_1} \mathbb{C}_q^N \otimes \dots \otimes \Lambda_q^{k_m} \mathbb{C}_q^N$ .
- On morphisms:

$$\begin{array}{c} k+l \\ \uparrow \\ \begin{array}{cc} \uparrow & \uparrow \\ k & l \end{array} \end{array} \mapsto m_{k,l}^{k+l}, \quad \begin{array}{c} k \quad l \\ \uparrow \quad \uparrow \\ \begin{array}{c} \uparrow \\ k+l \end{array} \end{array} \mapsto s_{k+l}^{k,l}$$

### Theorem (Cautis-Kamnitzer-Morrison 2012)

$\Gamma: N\text{-Web}_g^{\oplus} \rightarrow \mathfrak{sl}_N\text{-Mod}_e$  is an equivalence of (braided) monoidal categories.



# “Howe” to prove this?

Howe: the commuting actions of  $\mathbf{U}_q(\mathfrak{gl}_m)$  and  $\mathbf{U}_q(\mathfrak{sl}_N)$  on

$$\begin{aligned}\Lambda_q^K(\mathbb{C}_q^m \otimes \mathbb{C}_q^N) &\cong \bigoplus_{k_1+\dots+k_m=K} (\Lambda_q^{k_1}\mathbb{C}_q^N \otimes \dots \otimes \Lambda_q^{k_m}\mathbb{C}_q^N) \\ &\cong \bigoplus_{l_1+\dots+l_N=K} (\Lambda_q^{l_1}\mathbb{C}_q^m \otimes \dots \otimes \Lambda_q^{l_N}\mathbb{C}_q^m)\end{aligned}$$

introduce an  $\mathbf{U}_q(\mathfrak{gl}_m)$ -action  $f$  on the first term with  $\vec{k}$ -weight space  $\Lambda_q^{\vec{k}}\mathbb{C}_q^N$ .

In particular, there is a functorial action

$$\begin{aligned}\Phi_{\text{skew}}^m : \dot{\mathbf{U}}_q(\mathfrak{gl}_m) &\rightarrow \mathfrak{sl}_N\text{-Mod}_e, \\ \vec{k} \mapsto \Lambda_q^{\vec{k}}\mathbb{C}_q^N, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}} &\mapsto f(X) \in \text{Hom}_{\mathfrak{sl}_N\text{-Mod}_e}(\Lambda_q^{\vec{k}}\mathbb{C}_q^N, \Lambda_q^{\vec{l}}\mathbb{C}_q^N).\end{aligned}$$

Howe:  $\Phi_{\text{skew}}^m$  is full. Or in words: all relations in  $\mathfrak{sl}_N\text{-Mod}_e$  follow from the ones in  $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$  and the ones in the kernel of  $\Phi_{\text{skew}}^m$ .

# Define the diagrams to make this work

## Theorem(Cautis-Kamnitzer-Morrison 2012)

Define  $N\text{-Web}_g$  such there is a commutative diagram

$$\begin{array}{ccc}
 \dot{U}_q(\mathfrak{gl}_m) & \xrightarrow{\Phi_{\text{skew}}^m} & \mathfrak{sl}_N\text{-Mod}_e \\
 \searrow \Upsilon^m & & \nearrow \Gamma \\
 & N\text{-Web}_g &
 \end{array}$$

with

$$\Upsilon^m(F_i 1_{\vec{k}}) \mapsto \begin{array}{c} k_{i-1} \quad k_{i+1}+1 \\ \nearrow \quad \nearrow \\ \text{---} 1 \text{---} \\ \searrow \quad \searrow \\ k_i \quad k_{i+1} \end{array}, \quad \Upsilon^m(E_i 1_{\vec{k}}) \mapsto \begin{array}{c} k_i+1 \quad k_{i+1}-1 \\ \nearrow \quad \nearrow \\ \text{---} 1 \text{---} \\ \searrow \quad \searrow \\ k_i \quad k_{i+1} \end{array}$$

$\Upsilon^m$  induces the “ $\mathfrak{gl}_m$  ladder” relations,  $\ker(\Upsilon^m)$  gives the exterior relations.

# Exempli gratia

The “ $gl_m$  ladder” relation

$$\begin{array}{c} k \\ \uparrow \\ \text{---} \\ \uparrow \\ k-1 \\ \uparrow \\ \text{---} \\ \uparrow \\ k \end{array}
 \begin{array}{c} l \\ \uparrow \\ \text{---} \\ \uparrow \\ l+1 \\ \uparrow \\ \text{---} \\ \uparrow \\ l \end{array}
 -
 \begin{array}{c} k \\ \uparrow \\ \text{---} \\ \uparrow \\ k+1 \\ \uparrow \\ \text{---} \\ \uparrow \\ k \end{array}
 \begin{array}{c} l \\ \uparrow \\ \text{---} \\ \uparrow \\ l-1 \\ \uparrow \\ \text{---} \\ \uparrow \\ l \end{array}
 = [k - l]
 \begin{array}{c} k \\ | \\ k \end{array}
 \begin{array}{c} l \\ | \\ l \end{array}$$

is just

$$EF1_{\vec{k}} - FE1_{\vec{k}} = [k - l]1_{\vec{k}}.$$

The exterior relations are a diagrammatic version of

$$\Lambda_q^{>N} \mathbb{C}_q^N \cong 0.$$

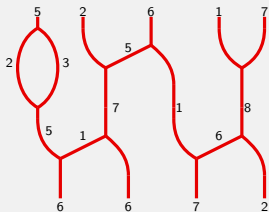
# The symmetric story is easier in some sense...

An  $2\text{-web}_T$  is a labeled, trivalent graph locally made of

$$\text{cap}_k = \begin{array}{c} \text{---} \text{---} \\ \text{ \textbackslash } / \\ \text{---} \end{array} \quad , \quad \text{cup}_k = \begin{array}{c} / \quad \backslash \\ \text{---} \end{array} \quad , \quad \text{m}_{k,l}^{k+l} = \begin{array}{c} \text{---} \\ | \\ \text{ \textbackslash } / \\ \text{---} \quad \text{---} \end{array} \quad , \quad \text{s}_{k+l}^{k,l} = \begin{array}{c} \text{---} \quad \text{---} \\ / \quad \backslash \\ \text{---} \end{array}$$

Up to sign issues that I ignore today!

## Example



# Never change a winning team

Define the (braided) monoidal,  $\mathbb{C}_q$ -linear category  $2\mathbf{Web}_r$  by using:

## Definition

Given  $\vec{k} \in \mathbb{Z}_{\geq 0}^n$  and  $\vec{l} \in \mathbb{Z}_{\geq 0}^{n'}$ . The 2-webs<sub>r</sub> space  $\text{Hom}_{2\mathbf{Web}_r}(\vec{k}, \vec{l})$  is the free  $\mathbb{C}_q$ -vector space generated by 2-webs<sub>r</sub> between  $\vec{k}$  and  $\vec{l}$  modulo isotopies and:

- The “gl<sub>n</sub> ladder” relations again!
- A circle evaluation and the *dumbbell relation*:

$$[2] \quad \begin{array}{|c|} \hline 1 \\ \hline \text{---} \\ \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline \text{---} \\ \hline 1 \\ \hline \end{array} = \begin{array}{c} \text{---} \\ \bigcup \\ \text{---} \\ \bigcap \\ \text{---} \\ \text{---} \end{array} + 2 \begin{array}{c} \text{---} \\ \bigcup \\ \text{---} \\ \text{---} \\ \bigcap \\ \text{---} \\ \text{---} \end{array}$$

The diagram shows the dumbbell relation. On the left, two vertical red lines represent the identity map on two strands. On the right, the identity is equal to the sum of two diagrams. The first diagram consists of two horizontal red lines with two arcs connecting them, one above and one below, forming a dumbbell shape. The second diagram consists of two vertical red lines on the left and right, and a vertical red line in the center. The top two strands are connected by an arc above the center line, and the bottom two strands are connected by an arc below the center line. A coefficient of 2 is placed to the left of this second diagram.

- But *no(!)* relation of the form

$$\begin{array}{|c|} \hline k \\ \hline \text{---} \\ \hline 1 \\ \hline \end{array} = 0 \quad , \quad \text{if } k > N.$$

The diagram shows a vertical red line with a coefficient  $k$  at the top and a  $1$  at the bottom. This is set equal to zero, with a comma and the condition “if  $k > N$ ”.

# Diagrams for intertwiners - Part 3

Observe that there are (up to scalars) unique  $\mathbf{U}_q(\mathfrak{sl}_2)$ -intertwiners

$$\text{cap}_k: \text{Sym}_q^k \mathbb{C}_q^2 \otimes \text{Sym}_q^k \mathbb{C}_q^2 \rightarrow \mathbb{C}_q \quad , \quad m_{k,l}^{k+l}: \text{Sym}_q^k \mathbb{C}_q^2 \otimes \text{Sym}_q^l \mathbb{C}_q^2 \rightarrow \text{Sym}_q^{k+l} \mathbb{C}_q^2$$

$$\text{cup}_k: \mathbb{C}_q \rightarrow \text{Sym}_q^k \mathbb{C}_q^2 \otimes \text{Sym}_q^k \mathbb{C}_q^2 \quad , \quad s_{k+l}^{k,l}: \text{Sym}_q^{k+l} \mathbb{C}_q^2 \rightarrow \text{Sym}_q^k \mathbb{C}_q^2 \otimes \text{Sym}_q^l \mathbb{C}_q^2$$

(guess where they come from...)

Let  $\mathfrak{sl}_2\text{-Mod}_s$  be the (braided) monoidal,  $\mathbb{C}_q$ -linear category whose objects are tensor generated by  $\text{Sym}_q^k \mathbb{C}_q^N$ . Define a functor  $\Gamma: 2\text{-Web}_r \rightarrow \mathfrak{sl}_2\text{-Mod}_s$ :

- On objects:  $\vec{k} = (k_1, \dots, k_n)$  is send to  $\text{Sym}_q^{k_1} \mathbb{C}_q^2 \otimes \dots \otimes \text{Sym}_q^{k_n} \mathbb{C}_q^2$ .
- On morphisms:

$$\text{cap}_k \mapsto \text{cap}_k \quad , \quad \text{cup}_k \mapsto \text{cup}_k \quad , \quad m_{k,l}^{k+l} \mapsto m_{k,l}^{k+l} \quad , \quad s_{k+l}^{k,l} \mapsto s_{k+l}^{k,l}$$

## Theorem

$\Gamma: 2\text{-Web}_r^\oplus \rightarrow \mathfrak{sl}_2\text{-Mod}_s$  is an equivalence of (braided) monoidal categories.

# “Howe” to prove this?

Howe: the commuting actions of  $\mathbf{U}_q(\mathfrak{gl}_n)$  and  $\mathbf{U}_q(\mathfrak{sl}_N)$  on

$$\begin{aligned}\mathrm{Sym}_q^K(\mathbb{C}_q^n \otimes \mathbb{C}_q^N) &\cong \bigoplus_{k_1+\dots+k_n=K} (\mathrm{Sym}_q^{k_1}\mathbb{C}_q^N \otimes \dots \otimes \mathrm{Sym}_q^{k_n}\mathbb{C}_q^N) \\ &\cong \bigoplus_{l_1+\dots+l_N=K} (\mathrm{Sym}_q^{l_1}\mathbb{C}_q^n \otimes \dots \otimes \mathrm{Sym}_q^{l_N}\mathbb{C}_q^n)\end{aligned}$$

introduce an  $\mathbf{U}_q(\mathfrak{gl}_n)$ -action  $f$  on the first term with  $\vec{k}$ -weight space  $\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^N$ .

In particular, there is a functorial action

$$\Phi_{\mathrm{sym}}^n : \dot{\mathbf{U}}_q(\mathfrak{gl}_n) \rightarrow \mathfrak{sl}_2\text{-Mod}_s,$$

$$\vec{k} \mapsto \mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_n)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{sl}_2\text{-Mod}_s}(\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \mathrm{Sym}_q^{\vec{l}}\mathbb{C}_q^2).$$

Howe:  $\Phi_{\mathrm{sym}}^n$  is full. Or in words: all relations in  $\mathfrak{sl}_2\text{-Mod}_s$  follow from the ones in  $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$  and the ones in the kernel of  $\Phi_{\mathrm{sym}}^n$ .

## Theorem

Define  $2\text{-Web}_r$  such that there is a commutative diagram

$$\begin{array}{ccc}
 \dot{U}_q(\mathfrak{gl}_n) & \xrightarrow{\Phi_{\text{sym}}^n} & \mathfrak{sl}_2\text{-Mod}_s \\
 \searrow \Upsilon^n & & \nearrow \Gamma \\
 & 2\text{-Web}_r &
 \end{array}$$

with

$$\Upsilon^n(F_i 1_{\vec{k}}) \mapsto \begin{array}{c} k-1 \quad l+1 \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ k \quad l \end{array}, \quad \Upsilon^n(E_i 1_{\vec{k}}) \mapsto \begin{array}{c} k+1 \quad l-1 \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ k \quad l \end{array}$$

$\Upsilon^n$  induces the “ $\mathfrak{gl}_n$  ladder” relations,  $\ker(\Upsilon^n)$  gives the circle/dumbbell relation.



# Exempli gratia

The dumbbell relation

$$[2] \begin{array}{|c|} \hline 1 \\ \hline | \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline | \\ \hline 1 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \cup \\ \hline 1 & 1 \\ \hline \end{array} + \begin{array}{|c|} \hline 1 & 1 \\ \hline \cup \\ \hline 2 \\ \hline \cup \\ \hline 1 & 1 \\ \hline \end{array}$$

is a diagrammatic version of

$$\mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \cong \mathbb{C}_q \oplus \text{Sym}_q^2 \mathbb{C}_q^2.$$

No relations of the form

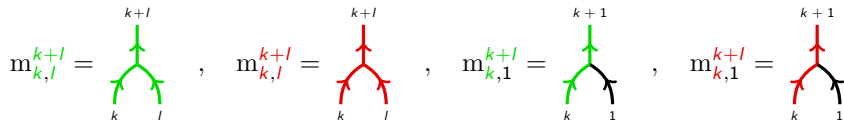
$$\left| \begin{array}{|c|} \hline k \\ \hline | \\ \hline \end{array} \right. = 0 \quad , \quad \text{if } k > N,$$

because

$$\text{Sym}_q^{>N} \mathbb{C}_q^N \neq 0.$$

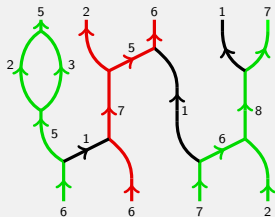
# Could there be a pattern?

An  $N$ -web<sub>gr</sub> is a colored, labeled, trivalent graph locally made of



And of course splits and some mirrors as well!

## Example



# The $N$ -webs<sub>gr</sub> category

Define the (braided) monoidal,  $\mathbb{C}_q$ -linear category  $N\text{-Web}_{\text{gr}}$  by using:

## Definition

Given  $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}$  and  $\vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$ . The  $N$ -webs<sub>gr</sub> space  $\text{Hom}_{N\text{-Web}_{\text{gr}}}(\vec{k}, \vec{l})$  is the free  $\mathbb{C}_q$ -vector space generated by  $N$ -webs<sub>gr</sub> between  $\vec{k}$  and  $\vec{l}$  modulo isotopies and:

- The “ $\mathfrak{gl}_m + \mathfrak{gl}_n$  ladder” relations.
- The dumbbell relation:

$$[2] \uparrow = \uparrow + \uparrow$$

- The exterior relations:

$$\uparrow_k = 0, \quad \text{if } k > N.$$

# Diagrams for intertwiners - Part 4

Observe that there are (up to scalars) unique  $\mathbf{U}_q(\mathfrak{sl}_N)$ -intertwiners

$$m_{k,1}^{k+1} : \Lambda_q^k \mathbb{C}_q^N \otimes \mathbb{C}_q^N \rightarrow \Lambda_q^{k+1} \mathbb{C}_q^N \quad \text{and} \quad m_{k,1}^{k+1} : \text{Sym}_q^k \mathbb{C}_q^N \otimes \mathbb{C}_q^N \rightarrow \text{Sym}_q^{k+1} \mathbb{C}_q^N$$

plus others as before.

Let  $\mathfrak{sl}_N\text{-Mod}_{\text{es}}$  be the (braided) monoidal,  $\mathbb{C}_q$ -linear category whose objects are tensor generated by  $\Lambda_q^k \mathbb{C}_q^N, \text{Sym}_q^k \mathbb{C}_q^N$ . Define a functor  $\Gamma : N\text{-Web}_{\text{gr}} \rightarrow \mathfrak{sl}_N\text{-Mod}_{\text{es}}$ :

- On objects:  $\vec{k} = (k_1, \dots, k_{m+n})$  is send to  $\Lambda_q^{k_1} \mathbb{C}_q^N \otimes \dots \otimes \text{Sym}_q^{k_{m+n}} \mathbb{C}_q^N$ .
- On morphisms:

$$\begin{array}{c} k+1 \\ \uparrow \\ \begin{array}{cc} \nearrow & \searrow \\ k & 1 \end{array} \end{array} \mapsto m_{k,1}^{k+1}, \quad \begin{array}{c} k+1 \\ \uparrow \\ \begin{array}{cc} \nearrow & \searrow \\ k & 1 \end{array} \end{array} \mapsto m_{k,1}^{k+1}, \quad \dots$$

## Theorem

$\Gamma : N\text{-Web}_{\text{gr}}^{\oplus} \rightarrow \mathfrak{sl}_N\text{-Mod}_{\text{es}}$  is an equivalence of (braided) monoidal categories.

## Definition

The *quantum general linear superalgebra*  $\mathbf{U}_q(\mathfrak{gl}(m|n))$  is generated by  $L_i^{\pm 1}$  and  $F_i, E_i$  subject to some relations, most notably, the *super relations*:

$$F_m^2 = 0 = E_m^2, \quad \frac{L_m L_{m+1}^{-1} - L_m^{-1} L_{m+1}}{q - q^{-1}} = F_m E_m + E_m F_m,$$

$$[2] F_m F_{m+1} F_{m-1} F_m = F_m F_{m+1} F_m F_{m-1} + F_{m-1} F_m F_{m+1} F_m \\ + F_{m+1} F_m F_{m-1} F_m + F_m F_{m-1} F_m F_{m+1} \text{ (plus an E version).}$$

There is a Howe pair  $(\mathbf{U}_q(\mathfrak{gl}(m|n)), \mathbf{U}_q(\mathfrak{sl}_N))$  with  $\vec{k} = (k_1, \dots, k_{m+n})$ -weight space under the  $\mathbf{U}_q(\mathfrak{gl}(m|n))$ -action on  $\Lambda_q^K(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^N)$  given by

$$\Lambda_q^{k_1} \mathbb{C}_q^N \otimes \cdots \otimes \Lambda_q^{k_m} \mathbb{C}_q^N \otimes \text{Sym}_q^{k_{m+1}} \mathbb{C}_q^N \otimes \cdots \otimes \text{Sym}_q^{k_{m+n}} \mathbb{C}_q^N.$$

# Define the diagrams to make this work

## Theorem

Define  $N\text{-Web}_{\text{gr}}$  such there is a commutative diagram

$$\begin{array}{ccc}
 \dot{U}_q(\mathfrak{gl}(m|n)) & \xrightarrow{\Phi_{\text{su}}^{m|n}} & \mathfrak{sl}_N\text{-Mod}_{\text{es}} \\
 \searrow \Upsilon_{\text{su}}^{m|n} & & \nearrow \Gamma \\
 & N\text{-Web}_{\text{gr}} &
 \end{array}$$

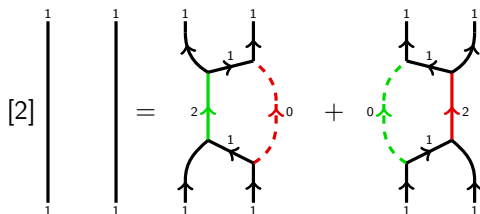
with

$$\Upsilon_{\text{su}}^{m|n}(F_m 1_{\vec{k}}) \mapsto \begin{array}{c} k_{m-1} \quad k_{m+1}+1 \\ \begin{array}{c} \text{green} \nearrow \quad \text{red} \nearrow \\ \text{black} \text{---} 1 \text{---} \\ \text{green} \searrow \quad \text{red} \searrow \\ k_m \quad k_{m+1} \end{array} \end{array}, \quad \Upsilon_{\text{su}}^{m|n}(E_m 1_{\vec{k}}) \mapsto \begin{array}{c} k_{m+1} \quad k_{m+1}-1 \\ \begin{array}{c} \text{green} \searrow \quad \text{red} \searrow \\ \text{black} \text{---} 1 \text{---} \\ \text{green} \nearrow \quad \text{red} \nearrow \\ k_m \quad k_{m+1} \end{array} \end{array}$$

$\Upsilon_{\text{su}}^{m|n}$  induces the “ $\mathfrak{gl}(m|n)$  ladder” relations,  $\ker(\Upsilon_{\text{su}}^{m|n})$  gives the exterior relations.

The dumbbell relation is the super commutator relation:

$$[2]1_{(1,1)} = F_m E_m 1_{(1,1)} + E_m F_m 1_{(1,1)}$$



$$\mathbb{C}_q^N \otimes \mathbb{C}_q^N \cong \Lambda_q^2 \mathbb{C}_q^N \oplus \text{Sym}_q^2 \mathbb{C}_q^N.$$

All other super relations are consequences!

# An almost perfect symmetry

Up to the exterior relations:  $N\text{-Web}_{\text{gr}}$  is completely symmetric in green-red. Only the *braiding* is slightly asymmetric, because  $q \leftrightarrow q^{-1}$ :

$$\begin{array}{c} \nearrow \\ \nwarrow \\ k \quad l \end{array} = (-1)^{k+kl} q^k \sum_{\substack{j_1, j_2 \geq 0 \\ j_1 - j_2 = k - l}} (-q)^{-j_1} \begin{array}{c} k - j_1 + j_2 \quad l + j_1 - j_2 \\ \nearrow \quad \nearrow \\ j_2 \\ \nwarrow \quad \nwarrow \\ k \quad l \\ \nearrow \quad \nearrow \\ j_1 \\ \nwarrow \quad \nwarrow \\ k \quad l \end{array}$$

$$\begin{array}{c} \nwarrow \\ \nearrow \\ k \quad l \end{array} = (-1)^k q^{-k} \sum_{\substack{j_1, j_2 \geq 0 \\ j_1 - j_2 = k - l}} (-q)^{+j_1} \begin{array}{c} k - j_1 + j_2 \quad l + j_1 - j_2 \\ \nwarrow \quad \nwarrow \\ j_2 \\ \nearrow \quad \nearrow \\ k \quad l \\ \nwarrow \quad \nwarrow \\ j_1 \\ \nearrow \quad \nearrow \\ k \quad l \end{array}$$



# The $\infty$ -webs $_{\text{gr}}$ space

Define as before  $\infty$ -**Web** $_{\text{gr}}$  by using:

## Definition

Given  $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}$  and  $\vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$ . The  $\infty$ -webs $_{\text{gr}}$  space  $\text{Hom}_{\infty\text{-Web}_{\text{gr}}}(\vec{k}, \vec{l})$  is the free  $\mathbb{C}_q$ -vector space generated by  $\infty$ -webs $_{\text{gr}}$  between  $\vec{k}, \vec{l}$  modulo isotopies and:

- The “ $\mathfrak{gl}_m + \mathfrak{gl}_n$  ladder” relations.
- The dumbbell relation:

$$[2] \begin{array}{c} 1 \\ | \\ \uparrow \\ | \\ 2 \\ | \\ \downarrow \\ | \\ 1 \end{array} = \begin{array}{c} 1 \quad 1 \\ \swarrow \quad \searrow \\ | \\ 2 \\ | \\ \swarrow \quad \searrow \\ 1 \quad 1 \end{array} + \begin{array}{c} 1 \quad 1 \\ \swarrow \quad \searrow \\ | \\ 2 \\ | \\ \swarrow \quad \searrow \\ 1 \quad 1 \end{array}$$

- But *no(!)* exterior relations!

$$\begin{array}{c} | \\ \uparrow \\ k \end{array} \neq 0, \quad \text{if } k > N.$$

# The “big category”

For all  $N \in \mathbb{N}$ : there is a commuting diagram

$$\begin{array}{ccc} \infty\text{-Web}_{\text{gr}} & \xrightarrow{\pi_{\infty}^N} & N\text{-Web}_{\text{gr}} \\ \Gamma^{\infty} \downarrow & & \downarrow \Gamma \\ \check{\mathbf{H}} & \xrightarrow{\pi^N} & \mathfrak{sl}_N\text{-Mod}_{\text{es}} \end{array}$$

Here  $\check{\mathbf{H}}$  is an *idempotented version of the Hecke algebra*  $\mathbf{H}$  and  $\pi^N$  is the full functor induced by  $q$ -Schur-Weyl duality:

$$\Phi_{q\text{SW}}^N : H_K(q) \xrightarrow{\cong} \text{End}_{\mathbf{U}_q(\mathfrak{sl}_N)}((\mathbb{C}_q^N)^{\otimes K}), \text{ if } N \geq K.$$

## Theorem

$\Gamma^{\infty} : \infty\text{-Web}_{\text{gr}}^{\oplus} \rightarrow \check{\mathbf{H}}$  is an equivalence of (braided) monoidal categories.

# An application: the HOMFLY-PT symmetry

Given a framed, oriented, colored knot  $\mathcal{K}$ , one can associate to it a polynomial called *colored HOMFLY-PT polynomial*  $\mathcal{P}_\lambda^{a,q}(\mathcal{K}) \in \mathbb{C}_q(a)$ . The colors  $\lambda$  are Young diagrams.

The colored HOMFLY-PT polynomial can be defined from  $\mathbf{H}$  and thus, from  $\infty\text{-Web}_{\text{gr}}$ . Since  $\infty\text{-Web}_{\text{gr}}$  is symmetric in green-red and the braiding is symmetric in green-red under  $q \leftrightarrow q^{-1}$ :

## Corollary(of the green $\leftrightarrow$ red symmetry)

The colored HOMFLY-PT polynomial satisfies

$$\mathcal{P}_\lambda^{a,q}(\mathcal{K}) = (-1)^{co} \mathcal{P}_{\lambda^T}^{a,q^{-1}}(\mathcal{K}),$$

where  $co$  is some constant.

There is still **much** to do...

Thanks for your attention!