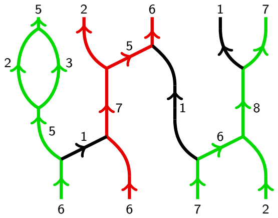
$\mathbf{U}_q(\mathfrak{sl}_N)$ diagram categories via q-Howe duality

Or: from dualities to diagrams

Daniel Tubbenhauer



Joint work with David Rose, Pedro Vaz and Paul Wedrich

March 2015

1. Exterior $\mathfrak{sl}_N$-web categories
   - Graphical calculus via Temperley-Lieb diagrams
   - Its cousins: the N-webs$_{\mathrm{g}}$
   - Proof? Skew quantum Howe duality!
2. Symmetric $\mathfrak{sl}_2$-web categories
   - More cousins: the 2-webs$_{\mathrm{r}}$
   - Proof? Symmetric quantum Howe duality!
3. Exterior-symmetric $\mathfrak{sl}_N$-web categories
   - Even more cousins: the N-webs$_{\mathrm{gr}}$
   - Proof? Super quantum Howe duality!
   - Green-red symmetry and the Hecke algebroid

The 2-web$_{\mathrm{g}}$ space

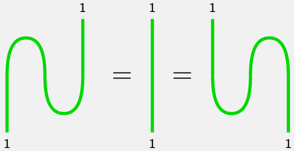
Definition(Rumer-Teller-Weyl 1932)

The 2-*web*$_{\mathrm{g}}$ *space* $\mathrm{Hom}_{2\text{-}\mathbf{Web}_{\mathrm{g}}}(b, t)$ is the free $\mathbb{C}_q = \mathbb{C}(q)$-vector space generated by non-intersecting arc diagrams with b, t bottom/top boundary points modulo:

- The *circle removal*:

$$= -q - q^{-1} = -[2]$$

- The *isotopy relations*:

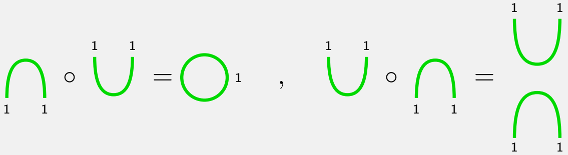


The 2-web$_{\mathrm{g}}$ category

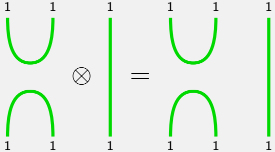
Definition(Kuperberg 1995)

The 2-*web*$_{\mathrm{g}}$ *category* 2-**Web**$_{\mathrm{g}}$ is the (braided) monoidal, $\mathbb{C}_q$-linear category with:

- Objects are vectors $\vec{k} = (1, \dots, 1)$ and morphisms are $\mathrm{Hom}_{2\text{-}\mathbf{Web}_{\mathrm{g}}}(\vec{k}, \vec{l})$.
- Composition $\circ$:



- Tensoring $\otimes$:



Diagrams for intertwiners

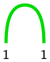
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners

$$\mathrm{cap}\colon \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \to \mathbb{C}_q \quad \text{and} \quad \mathrm{cup}\colon \mathbb{C}_q \to \mathbb{C}_q^2 \otimes \mathbb{C}_q^2,$$

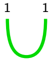
projecting $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$ onto $\mathbb{C}_q$ respectively embedding $\mathbb{C}_q$ into $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$.

Let $\mathfrak{sl}_2$-$\mathbf{Mod}_e$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\mathbb{C}_q^2$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{g}} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_e$:

- On objects: $\vec{k} = (1, \dots, 1)$ is send to $(\mathbb{C}_q^2)^{\otimes k} = \mathbb{C}_q^2 \otimes \cdots \otimes \mathbb{C}_q^2$.
- On morphisms:



$\mapsto \mathrm{cap}$,



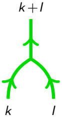
$\mapsto \mathrm{cup}$

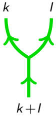
Theorem(Folklore)

$\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{g}}^{\oplus} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_e$ is an equivalence of (braided) monoidal categories.

The main step beyond $\mathfrak{sl}_2$: trivalent vertices

An N-*web*$_{\mathfrak{g}}$ is an oriented, labeled, trivalent graph locally made of

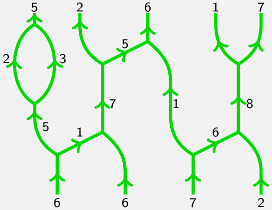
$\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{s}_{k+l}^{k,l} =$ 

$k, l, k+l \in \mathbb{N}$

(and no pivotal things today).

Example



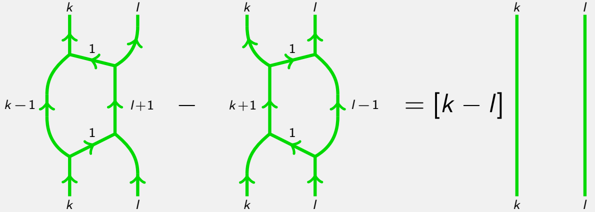
Let us try the same for $\mathfrak{sl}_N$: the N-web$_{\mathrm{g}}$ space

Define the (braided) monoidal, $\mathbb{C}_q$-linear category N-$\mathbf{Web}_{\mathrm{g}}$ by using:

Definition(Cautis-Kamnitzer-Morrison 2012)

The *N-web$_{\mathrm{g}}$ space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathrm{g}}}(\vec{k},\vec{l})$ is the free $\mathbb{C}_q$-vector space generated by N-webs$_{\mathrm{g}}$ with $\vec{k}$ and $\vec{l}$ at the bottom and top modulo isotopies and:

- "$\mathfrak{gl}_m$ *ladder*" relations like



- The *exterior relations*:

$= 0 \quad , \quad$ if $k > N$.

Diagrams for intertwiners - Part 2

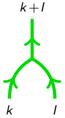
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_N)$-intertwiners

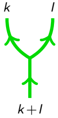
$$\mathrm{m}_{k,l}^{k+l}\colon \Lambda_q^k\mathbb{C}_q^N \otimes \Lambda_q^l\mathbb{C}_q^N \to \Lambda_q^{k+l}\mathbb{C}_q^N \quad \text{and} \quad \mathrm{s}_{k+l}^{k,l}\colon \Lambda_q^{k+l}\mathbb{C}_q^N \to \Lambda_q^k\mathbb{C}_q^N \otimes \Lambda_q^l\mathbb{C}_q^N$$

given by projection and inclusion again.

Let $\mathfrak{sl}_N$-$\mathbf{Mod}_e$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\Lambda_q^k\mathbb{C}_q^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}} \to \mathfrak{sl}_N\text{-}\mathbf{Mod}_e$:

- On objects: $\vec{k} = (k_1, \dots, k_m)$ is send to $\Lambda_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \Lambda_q^{k_m}\mathbb{C}_q^N$.
- On morphisms:



$\mapsto \mathrm{m}_{k,l}^{k+l}$,


$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem(Cautis-Kamnitzer-Morrison 2012)

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}}^{\oplus} \to \mathfrak{sl}_N\text{-}\mathbf{Mod}_e$ is an equivalence of (braided) monoidal categories.

"Howe" to prove this?

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_m)$ and $\mathbf{U}_q(\mathfrak{sl}_N)$ on

$$\textstyle\bigwedge_q^K(\mathbb{C}_q^m \otimes \mathbb{C}_q^N) \cong \bigoplus_{k_1+\cdots+k_m=K} (\bigwedge_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \bigwedge_q^{k_m}\mathbb{C}_q^N) \\ \cong \bigoplus_{l_1+\cdots+l_N=K} (\bigwedge_q^{l_1}\mathbb{C}_q^m \otimes \cdots \otimes \bigwedge_q^{l_N}\mathbb{C}_q^m)$$

introduce an $\mathbf{U}_q(\mathfrak{gl}_m)$-action f on the first term with $\vec{k}$-weight space $\bigwedge_q^{\vec{k}}\mathbb{C}_q^N$.

In particular, there is a functorial action

$$\Phi^m_{\mathrm{skew}} \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{sl}_N\text{-}\mathbf{Mod}_e,$$
$$\vec{k} \mapsto \textstyle\bigwedge_q^{\vec{k}}\mathbb{C}_q^N, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{sl}_N\text{-}\mathbf{Mod}_e}(\bigwedge_q^{\vec{k}}\mathbb{C}_q^N, \bigwedge_q^{\vec{l}}\mathbb{C}_q^N).$$

Howe: Φ^m_{skew} is full. Or in words: all relations in $\mathfrak{sl}_N$-$\mathbf{Mod}_e$ follow from the ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ and the ones in the kernel of Φ^m_{skew}.

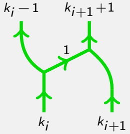
Define the diagrams to make this work

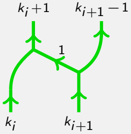
Theorem(Cautis-Kamnitzer-Morrison 2012)

Define N-$\mathbf{Web}_{\mathrm{g}}$ such there is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_q(\mathfrak{gl}_m) & \xrightarrow{\Phi^m_{\mathrm{skew}}} & \mathfrak{sl}_N\text{-}\mathbf{Mod}_e \\ & {\scriptstyle \Upsilon^m}\searrow \quad \nearrow{\scriptstyle \Gamma} & \\ & N\text{-}\mathbf{Web}_{\mathrm{g}} & \end{array}$$

with

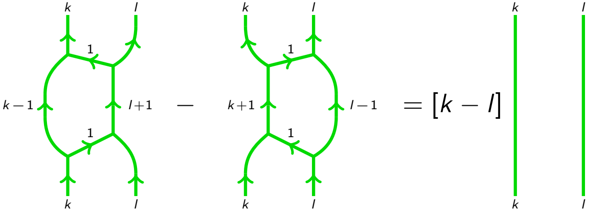
$\Upsilon^m(F_i 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^m(E_i 1_{\vec{k}}) \mapsto$ 

Υ^m induces the "$\mathfrak{gl}_m$ ladder" relations, $\ker(\Upsilon^m)$ gives the exterior relations.

Exempli gratia

The "$\mathfrak{gl}_m$ ladder" relation



is just

$$EF1_{\vec{k}} - FE1_{\vec{k}} = [k-l]1_{\vec{k}}.$$

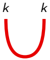
The exterior relations are a diagrammatic version of

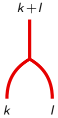
$$\textstyle\bigwedge_q^{>N}\mathbb{C}_q^N \cong 0.$$

The symmetric story is easier in some sense...

An 2-*web*$_{\mathrm{r}}$ is a labeled, trivalent graph locally made of

$\mathrm{cap}_k =$ 

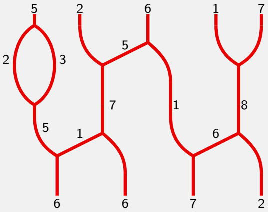
, $\mathrm{cup}_k =$ 

, $\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{s}_{k+l}^{k,l} =$ 

Up to sign issues that I ignore today!

Example



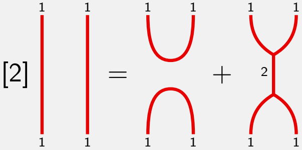
Never change a winning team

Define the (braided) monoidal, $\mathbb{C}_q$-linear category 2-$\mathbf{Web}_{\mathrm{r}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}^{n}_{\geq 0}$ and $\vec{l} \in \mathbb{Z}^{n'}_{\geq 0}$. The 2-*webs*$_{\mathrm{r}}$ *space* $\mathrm{Hom}_{2\text{-}\mathbf{Web}_{\mathrm{r}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by 2-webs$_{\mathrm{r}}$ between $\vec{k}$ and $\vec{l}$ modulo isotopies and:

- The "$\mathfrak{gl}_n$ *ladder*" relations again!
- A circle evaluation and the *dumbbell relation*:



- But *no(!)* relation of the form

$= 0 \quad , \quad \text{if } k > N.$

Diagrams for intertwiners - Part 3

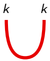
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners

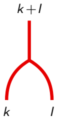
$$\mathrm{cap}_k\colon \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^k\mathbb{C}_q^2 \to \mathbb{C}_q \quad , \quad \mathrm{m}_{k,l}^{k+l}\colon \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2 \to \mathrm{Sym}_q^{k+l}\mathbb{C}_q^2$$
$$\mathrm{cup}_k\colon \mathbb{C}_q \to \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^k\mathbb{C}_q^2 \quad , \quad \mathrm{s}_{k+l}^{k,l}\colon \mathrm{Sym}_q^{k+l}\mathbb{C}_q^2 \to \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2$$

(guess where they come from...)

Let $\mathfrak{sl}_2$-$\mathbf{Mod}_s$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\mathrm{Sym}_q^k\mathbb{C}_q^N$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{r}} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_s$:

- On objects: $\vec{k} = (k_1, \dots, k_n)$ is send to $\mathrm{Sym}_q^{k_1}\mathbb{C}_q^2 \otimes \cdots \otimes \mathrm{Sym}_q^{k_n}\mathbb{C}_q^2$.
- On morphisms:

$\mapsto \mathrm{cap}_k$,


$\mapsto \mathrm{cup}_k$,


$\mapsto \mathrm{m}_{k,l}^{k+l}$,


$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem

$\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{r}}^{\oplus} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_s$ is an equivalence of (braided) monoidal categories.

“Howe” to prove this?

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_N)$ on

$$\mathrm{Sym}_q^K(\mathbb{C}_q^n \otimes \mathbb{C}_q^N) \cong \bigoplus_{k_1+\cdots+k_n=K} (\mathrm{Sym}_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_n}\mathbb{C}_q^N)$$
$$\cong \bigoplus_{l_1+\cdots+l_N=K} (\mathrm{Sym}_q^{l_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{l_N}\mathbb{C}_q^n)$$

introduce an $\mathbf{U}_q(\mathfrak{gl}_n)$-action f on the first term with $\vec{k}$-weight space $\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^N$.

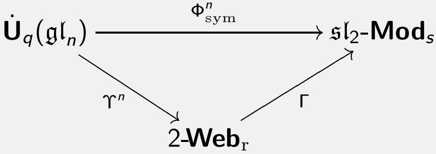
In particular, there is a functorial action

$$\Phi^n_{\mathrm{sym}}\colon \dot{\mathbf{U}}_q(\mathfrak{gl}_n) \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_s,$$
$$\vec{k} \mapsto \mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_n)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{sl}_2\text{-}\mathbf{Mod}_s}(\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \mathrm{Sym}_q^{\vec{l}}\mathbb{C}_q^2).$$

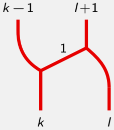
Howe: Φ^n_{sym} is full. Or in words: all relations in $\mathfrak{sl}_2$-$\mathbf{Mod}_s$ follow from the ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$ and the ones in the kernel of Φ^n_{sym}.

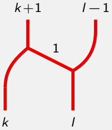
Theorem

Define 2-$\mathbf{Web}_{\mathrm{r}}$ such that there is a commutative diagram



with

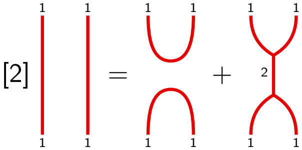
$\Upsilon^n(F_i 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^n(E_i 1_{\vec{k}}) \mapsto$ 

Υ^n induces the "$\mathfrak{gl}_n$ ladder" relations, $\ker(\Upsilon^n)$ gives the circle/dumbbell relation.

Exempli gratia

The dumbbell relation



is a diagrammatic version of

$$\mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \cong \mathbb{C}_q \oplus \mathrm{Sym}_q^2 \mathbb{C}_q^2.$$

No relations of the form

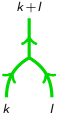
$$= 0 \quad , \quad \text{if } k > N,$$

because

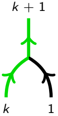
$$\mathrm{Sym}_q^{>N} \mathbb{C}_q^N \ncong 0.$$

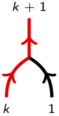
Could there be a pattern?

An N-web$_{\mathrm{gr}}$ is a colored, labeled, trivalent graph locally made of

$\mathrm{m}_{k,l}^{k+l} =$ 

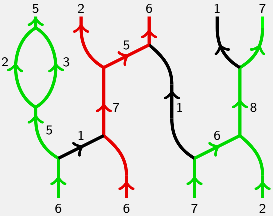
, $\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{m}_{k,1}^{k+l} =$ 

, $\mathrm{m}_{k,1}^{k+l} =$ 

And of course splits and some mirrors as well!

Example



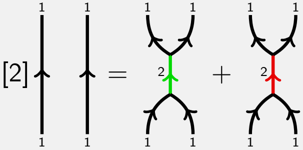
The N-webs$_{\mathrm{gr}}$ category

Define the (braided) monoidal, $\mathbb{C}_q$-linear category N-$\mathbf{Web}_{\mathrm{gr}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}$ and $\vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$. The *$N$-webs$_{\mathrm{gr}}$ space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathrm{gr}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by N-webs$_{\mathrm{gr}}$ between $\vec{k}$ and $\vec{l}$ modulo isotopies and:

- The "$\mathfrak{gl}_m$+$\mathfrak{gl}_n$ ladder" relations.
- The dumbbell relation:



- The *exterior relations*:

$= 0 \quad , \quad \text{if } k > N.$

Diagrams for intertwiners - Part 4

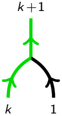
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_N)$-intertwiners

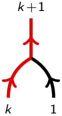
$$\mathrm{m}_{k,1}^{k+1}\colon \textstyle\bigwedge_q^k \mathbb{C}_q^N \otimes \mathbb{C}_q^N \to \textstyle\bigwedge_q^{k+1}\mathbb{C}_q^N \quad \text{and} \quad \mathrm{m}_{k,1}^{k+1}\colon \mathrm{Sym}_q^k \mathbb{C}_q^N \otimes \mathbb{C}_q^N \to \mathrm{Sym}_q^{k+1}\mathbb{C}_q^N$$

plus others as before.

Let $\mathfrak{sl}_N$-$\mathbf{Mod}_{\mathrm{es}}$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\bigwedge_q^k \mathbb{C}_q^N, \mathrm{Sym}_q^k \mathbb{C}_q^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}} \to \mathfrak{sl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$:

- On objects: $\vec{k} = (k_1, \dots, k_{m+n})$ is send to $\bigwedge_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_{m+n}}\mathbb{C}_q^N$.
- On morphisms:



$\mapsto \mathrm{m}_{k,1}^{k+1}$,


$\mapsto \mathrm{m}_{k,1}^{k+1}$, $\cdots$

Theorem

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}}^{\oplus} \to \mathfrak{sl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$ is an equivalence of (braided) monoidal categories.

Super $\mathfrak{gl}(m|n)$

Definition

The *quantum general linear superalgebra* $\mathbf{U}_q(\mathfrak{gl}(m|n))$ is generated by $L_i^{\pm 1}$ and F_i, E_i subject the some relations, most notably, the *super relations*:

$$F_m^2 = 0 = E_m^2 \quad , \quad \frac{L_m L_{m+1}^{-1} - L_m^{-1} L_{m+1}}{q - q^{-1}} = F_m E_m + E_m F_m,$$

$$\begin{aligned}[2] F_m F_{m+1} F_{m-1} F_m = & F_m F_{m+1} F_m F_{m-1} + F_{m-1} F_m F_{m+1} F_m \\ & + F_{m+1} F_m F_{m-1} F_m + F_m F_{m-1} F_m F_{m+1} \text{ (plus an E version).}\end{aligned}$$

There is a Howe pair $(\mathbf{U}_q(\mathfrak{gl}(m|n)), \mathbf{U}_q(\mathfrak{sl}_N))$ with $\vec{k} = (k_1, \dots, k_{m+n})$-weight space under the $\mathbf{U}_q(\mathfrak{gl}(m|n))$-action on $\bigwedge_q^K(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^N)$ given by

$$\textstyle\bigwedge_q^{k_1} \mathbb{C}_q^N \otimes \cdots \bigwedge_q^{k_m} \mathbb{C}_q^N \otimes \mathrm{Sym}_q^{k_{m+1}} \mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_{m+n}} \mathbb{C}_q^N.$$

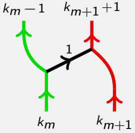
Define the diagrams to make this work

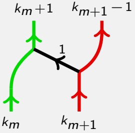
Theorem

Define N-$\mathbf{Web}_{\mathrm{gr}}$ such there is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_q(\mathfrak{gl}(m|n)) & \xrightarrow{\Phi^{m|n}_{\mathrm{su}}} & \mathfrak{sl}_N\text{-}\mathbf{Mod}_{\mathrm{es}} \\ & {\scriptstyle \Upsilon^{m|n}_{\mathrm{su}}} \searrow \quad \nearrow {\scriptstyle \Gamma} & \\ & N\text{-}\mathbf{Web}_{\mathrm{gr}} & \end{array}$$

with

$\Upsilon^{m|n}_{\mathrm{su}}(F_m 1_{\vec{k}}) \mapsto$ 

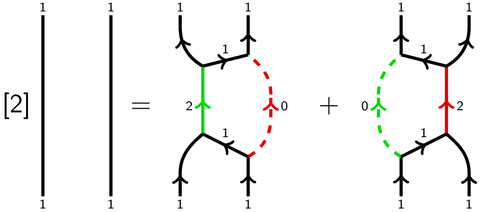
, $\Upsilon^{m|n}_{\mathrm{su}}(E_m 1_{\vec{k}}) \mapsto$ 

$\Upsilon^{m|n}_{\mathrm{su}}$ induces the "$\mathfrak{gl}(m|n)$ ladder" relations, $\ker(\Upsilon^{m|n}_{\mathrm{su}})$ gives the exterior relations.

Exempli gratia

The dumbbell relation is the super commutator relation:

$$[2]1_{(1,1)} = F_m E_m 1_{(1,1)} + E_m F_m 1_{(1,1)}$$



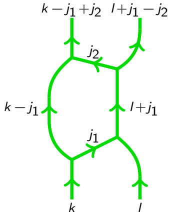
$$\mathbb{C}_q^N \otimes \mathbb{C}_q^N \cong \Lambda_q^2 \mathbb{C}_q^N \oplus \mathrm{Sym}_q^2 \mathbb{C}_q^N.$$

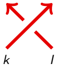
All other super relations are consequences!

An almost perfect symmetry

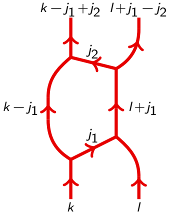
Up to the exterior relations: N-$\mathbf{Web}_{\mathrm{gr}}$ is completely symmetric in green-red. Only the *braiding* is slightly asymmetric, because $q \leftrightarrow q^{-1}$:

$$= (-1)^{k+kl} q^{k} \sum_{\substack{j_1, j_2 \geq 0 \\ j_1 - j_2 = k - l}} (-q)^{-j_1}$$





$$= (-1)^{k} q^{-k} \sum_{\substack{j_1, j_2 \geq 0 \\ j_1 - j_2 = k - l}} (-q)^{+j_1}$$



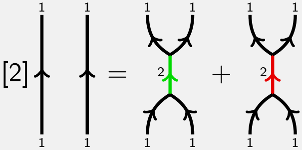
The ∞-webs$_{\mathrm{gr}}$ space

Define as before ∞-**Web**$_{\mathrm{gr}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}$ and $\vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$. The *$\infty$-webs$_{\mathrm{gr}}$ space* $\mathrm{Hom}_{\infty\text{-}\mathbf{Web}_{\mathrm{gr}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by ∞-webs$_{\mathrm{gr}}$ between $\vec{k}, \vec{l}$ modulo isotopies and:

- The "$\mathfrak{gl}_m + \mathfrak{gl}_n$ ladder" relations.
- The dumbbell relation:



- But *no(!)* exterior relations!

 $k \neq 0$, if $k > N$.

The "big category"

For all $N \in \mathbb{N}$: there is a commuting diagram

$$\begin{array}{ccc} \infty\text{-}\mathbf{Web}_{\mathrm{gr}} & \xrightarrow{\pi^N_\infty} & N\text{-}\mathbf{Web}_{\mathrm{gr}} \\ \downarrow{\scriptstyle \Gamma^\infty} & & \downarrow{\scriptstyle \Gamma} \\ \check{\mathbf{H}} & \xrightarrow[\pi^N]{} & \mathfrak{sl}_N\text{-}\mathbf{Mod}_{\mathrm{es}} \end{array}$$

Here $\check{\mathbf{H}}$ is an *idempotented version of the Hecke algebroid* $\mathbf{H}$ and π^N is the full functor induced by *q-Schur-Weyl duality*:

$$\Phi^N_{q\mathrm{SW}} \colon H_K(q) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_N)}((\mathbb{C}^N_q)^{\otimes K}), \text{ if } N \geq K.$$

Theorem

$\Gamma^\infty \colon \infty\text{-}\mathbf{Web}^{\oplus}_{\mathrm{gr}} \to \check{\mathbf{H}}$ is an equivalence of (braided) monoidal categories.

An application: the HOMFLY-PT symmetry

Given a framed, oriented, colored knot $\mathcal{K}$, one can associate to it a polynomial called *colored HOMFLY-PT polynomial* $\mathcal{P}_{\lambda}^{a,q}(\mathcal{K}) \in \mathbb{C}_q(a)$. The colors λ are Young diagrams.

The colored HOMFLY-PT polynomial can be defined from **H** and thus, from ∞-**Web**$_{\mathrm{gr}}$. Since ∞-**Web**$_{\mathrm{gr}}$ is symmetric in green-red and the braiding is symmetric in green-red under $q \leftrightarrow q^{-1}$:

Corollary(of the green$\leftrightarrow$red symmetry)

The colored HOMFLY-PT polynomial satisfies

$$\mathcal{P}_{\lambda}^{a,q}(\mathcal{K}) = (-1)^{co}\mathcal{P}_{\lambda^{\mathrm{T}}}^{a,q^{-1}}(\mathcal{K}),$$

where *co* is some constant.

There is still much to do...

Thanks for your attention!