(Singular) TQFTs, link homologies and Lie theory 2

Or: fun with singular surfaces

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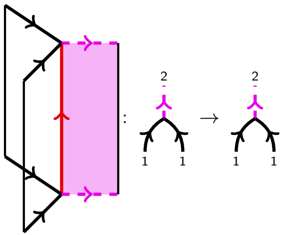


Joint work (still in progress) with Michael Ehrig and Catharina Stroppel

November 2015

1. Why foams? They categorify intertwiners!
   - $\mathfrak{sl}_2$-webs
   - $\mathfrak{gl}_2$-webs
   - Further generalizations

2. Singular TQFTs and foams
   - What are foams? The informal answer
   - What are foams? The singular TQFT construction
   - The web algebra

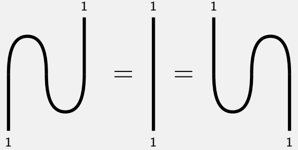
Definition(Rumer-Teller-Weyl 1932)

The 2-web space $\mathrm{Hom}_{\textbf{2-Web}}(b, t)$ is the free $\mathbb{C}(q)$-vector space generated by non-intersecting arc diagrams with b, t bottom/top boundary points modulo:

Circle removal : 

$= (q + q^{-1}) = [2].$

Isotopy relations :

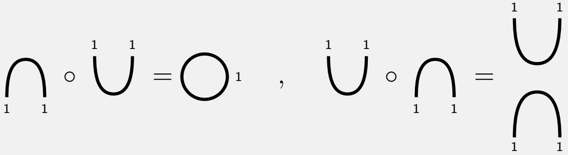


The 2-web category

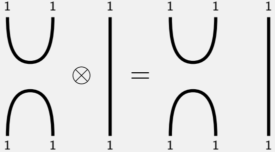
Definition(Kuperberg 1995)

The 2-web category 2-**Web** is the (braided) monoidal, $\mathbb{C}(q)$-linear category with:

- Objects are vectors $\vec{k} = (1, \ldots, 1)$ and morphisms are $\mathrm{Hom}_{2\text{-}\mathbf{Web}}(\vec{k}, \vec{l})$.
- Composition $\circ$:



- Tensoring $\otimes$:



Diagrams for intertwiners

Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners

$$\mathrm{cap}\colon \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \twoheadrightarrow \mathbb{C}(q), \quad \mathrm{cup}\colon \mathbb{C}(q) \hookrightarrow \mathbb{C}_q^2 \otimes \mathbb{C}_q^2,$$

projecting $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$ onto $\mathbb{C}(q)$ respectively embedding $\mathbb{C}(q)$ into $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$.

Let $\mathfrak{sl}_2$-**Mod** be the (braided) monoidal, $\mathbb{C}(q)$-linear category whose objects are tensor generated by $\mathbb{C}_q^2$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}$:

$$\vec{k} = (1, \dots, 1) \mapsto \mathbb{C}_q^2 \otimes \cdots \otimes \mathbb{C}_q^2,$$

$\mapsto \mathrm{cap}$,

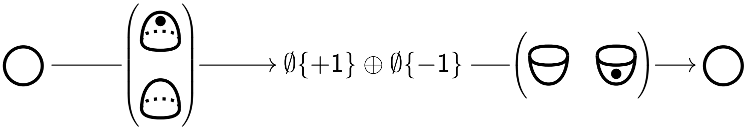
$\mapsto \mathrm{cup}$

Theorem(Folklore)

$\Gamma\colon 2\text{-}\mathbf{Web}^{\oplus} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}$ is an equivalence of (braided) monoidal categories.

Categorification "adds one dimension"

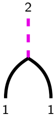
- Each generic slice of the cobordisms from 2-$\mathbf{Cob}$ is a 2-web.
- In fact, one can see 2-$\mathbf{Cob}_{\mathbb{C}}$ as a 2-category that categorifies 2-**Web** (in a suitable sense). In particular, the relations in 2-**Web** are lifted to equivalences of 1-morphisms in 2-$\mathbf{Cob}_{\mathbb{C}}$:

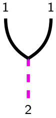


- By the representation theorem and Reshetikhin-Turaev's construction: the category 2-**Web** can be used to calculate the Jones polynomial.
- Thus, 2-$\mathbf{Cob}$ should give Khovanov homology - and indeed, it does.

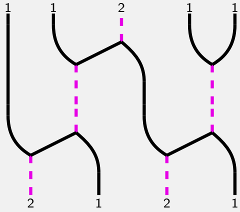
Webs - revised

A $\mathfrak{gl}_2$-web is a labeled trivalent graph locally made of

m = 

, s = 

Example



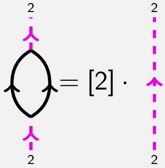
Let us form a category again

Define the (braided) monoidal, $\mathbb{C}(q)$-linear category **2-reWeb** by using:

Definition

The revised 2-web space $\mathrm{Hom}_{\mathbf{2\text{-}reWeb}}(\vec{k},\vec{l})$ with $\vec{k},\vec{l}\in\{0,1,2\}^{\mathbb{Z}}$ is the free $\mathbb{C}(q)$-vector space generated by $\mathfrak{gl}_2$-webs modulo the "circle" removal

"Circle" removal:



and isotopies fixing the boundary.

Diagrams for intertwiners

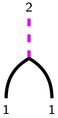
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_2)$-intertwiners

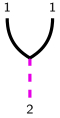
$$\mathrm{m}\colon \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \twoheadrightarrow \Lambda_q^2\mathbb{C}_q^2, \quad \mathrm{s}\colon \Lambda_q^2\mathbb{C}_q^2 \hookrightarrow \mathbb{C}_q^2 \otimes \mathbb{C}_q^2$$

given by projection and inclusion.

Let $\mathfrak{gl}_2$-**Mod** be the (braided) monoidal, $\mathbb{C}(q)$-linear category whose objects are tensor generated by $\mathbb{C}_q^2$ and $\Lambda_q^2\mathbb{C}_q^2$. Define a functor $\Gamma\colon$ 2-**reWeb** $\to \mathfrak{gl}_2$-**Mod**:

$$\vec{k} = (0,1,1,2,0) \mapsto \mathbb{C}(q) \otimes \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \otimes \Lambda_q^2\mathbb{C}_q^2 \otimes \mathbb{C}(q), \text{ etc.}$$



$\mapsto \mathrm{m}$,


$\mapsto \mathrm{s}$

Theorem

$\Gamma\colon$ 2-**reWeb**$^{\oplus} \to \mathfrak{gl}_2$-**Mod** is an equivalence of (braided) monoidal categories.

From $\mathfrak{gl}_2$ to $\mathfrak{sl}_2$

Restricting from $\mathfrak{gl}_2$ to $\mathfrak{sl}_2$ could increase the number of intertwiners:

$$\mathbf{U}_q(\mathfrak{sl}_2) \subset \mathbf{U}_q(\mathfrak{gl}_2) \quad \Rightarrow \quad \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{sl}_2)}(M, M') \supset \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{gl}_2)}(M, M').$$

Note that $\mathbb{C}_q^2$ is self-dual as a $\mathbf{U}_q(\mathfrak{sl}_2)$-module, but not as a $\mathbf{U}_q(\mathfrak{gl}_2)$-module. We obtain extra diagrams:

$: \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \to \mathbb{C}(q),$ 

$: \mathbb{C}(q) \to \mathbb{C}_q^2 \otimes \mathbb{C}_q^2.$

In particular, the so-called determinant representation $\Lambda_q^2\mathbb{C}_q^2$ satisfies

$$\Lambda_q^2\mathbb{C}_q^2 \cong \mathbb{C}(q) \quad \text{as } \mathbf{U}_q(\mathfrak{sl}_2)\text{-modules},$$
$$\Lambda_q^2\mathbb{C}_q^2 \ncong \mathbb{C}(q) \quad \text{as } \mathbf{U}_q(\mathfrak{gl}_2)\text{-modules}.$$

Diagrams for intertwiners

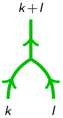
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_M)$-intertwiners

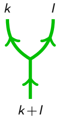
$$\mathrm{m}_{k,l}^{k+l}\colon \Lambda_q^k\mathbb{C}_q^M \otimes \Lambda_q^l\mathbb{C}_q^M \twoheadrightarrow \Lambda_q^{k+l}\mathbb{C}_q^M \quad \text{and} \quad \mathrm{s}_{k+l}^{k,l}\colon \Lambda_q^{k+l}\mathbb{C}_q^M \hookrightarrow \Lambda_q^k\mathbb{C}_q^M \otimes \Lambda_q^l\mathbb{C}_q^M$$

given by projection and inclusion.

Let $\mathfrak{gl}_M$-$\mathbf{Mod}_e$ be the (braided) monoidal, $\mathbb{C}(q)$-linear category whose objects are tensor generated by $\Lambda_q^k\mathbb{C}_q^M$. Define a functor $\Gamma\colon M\text{-}\mathbf{Web}_{\mathrm{g}} \to \mathfrak{gl}_M\text{-}\mathbf{Mod}_e$:

$$\vec{k} = (k_1,\dots,k_m) \mapsto \Lambda_q^{k_1}\mathbb{C}_q^M \otimes \cdots \otimes \Lambda_q^{k_m}\mathbb{C}_q^M,$$



$\mapsto \mathrm{m}_{k,l}^{k+l}$,


$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem(Cautis-Kamnitzer-Morrison 2012)

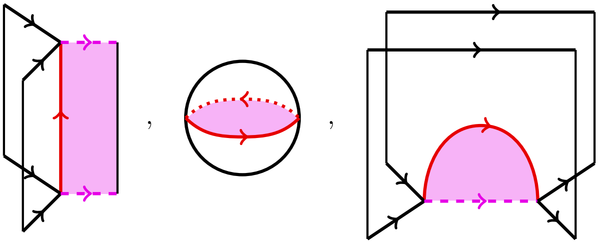
$\Gamma\colon M\text{-}\mathbf{Web}_{\mathrm{g}}^{\oplus} \to \mathfrak{gl}_M\text{-}\mathbf{Mod}_e$ is an equivalence of (braided) monoidal categories.

Categorification still "adds one dimension"

- The same pattern continues for other categories of intertwiners: one always needs trivalent vertices.
- If we believe in Khovanov's categorification approach using (and we do), then we should find a "cobordism" category whose generic slices are trivalent graphs aka webs.
- These are foams! We cook them up using singular TQFTs where the singular seams "categorify" the trivalent vertex.
- Note that the sign issue for the functoriality of Khovanov homology roughly comes from the identification of $\Lambda^2_q\mathbb{C}^2_q$ with the trivial module.

Foams in a nutshell

Informally, a $\mathfrak{gl}_2$-foam is a two-dimensional CW-complex with singular circles, some additional data and modulo some relations. A point on a singular circle has a neighborhood homeomorphic to the product of the letter Y and an interval



Here generic slices are $\mathfrak{sl}_2$-webs!

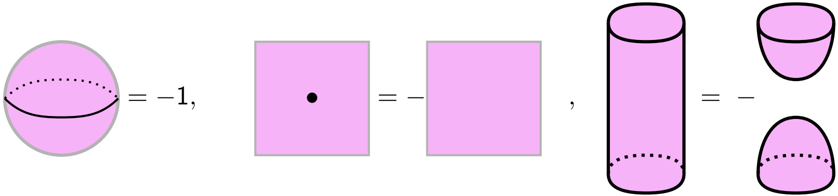
"Usual" TQFTs

Recall that equivalence classes of TQFTs for surfaces are in one-to-one correspondence with isomorphism classes of finite-dimensional, commutative Frobenius algebras. The Frobenius algebras we need are

$$\mathcal{A}_1 = \mathbb{C}[X]/(X^2), \qquad \mathcal{A}_2 = \mathbb{C},$$

with a non-trivial trace $\mathrm{tr}_2(1) = -1$ for the second.

We have seen the TQFT for $\mathcal{A}_1$ before. The one for $\mathcal{A}_2$ has relations like

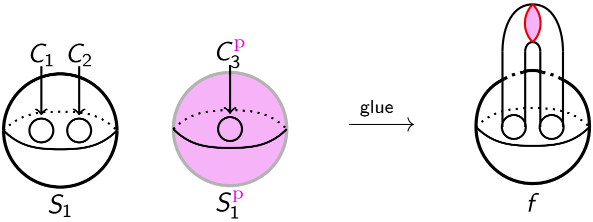


Singular surfaces

Fix the following data denoted by **S**:

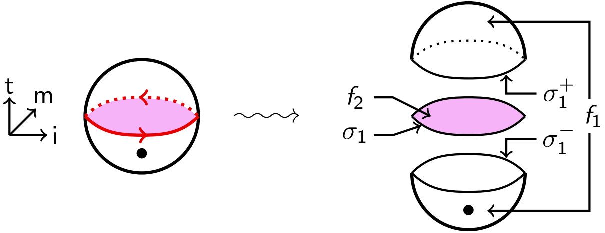
1. A surface S with connected components divided into two sets $S_1, \dots, S_r$ and $S_1^{\mathrm{p}}, \dots, S_{r'}^{\mathrm{p}}$, called ordinary surfaces and phantom surfaces.
2. The boundary components of S are partitioned into triples $(C_i, C_j, C_k^{\mathrm{p}})$ such that each triple contains precisely one phantom boundary component.
3. The three circles C_i, C_j and C_k^{p} in each triple are identified via diffeomorphisms $\varphi_{ij}\colon C_i \to C_j$ and $\varphi_{jk}\colon C_j \to C_k^{\mathrm{p}}$.
4. A finite (possible empty) set of "dots" per connected components $S_1, \dots, S_r$ and $S_1^{\mathrm{p}}, \dots, S_{r'}^{\mathrm{p}}$ that move freely around its connected component.

Now identify via $\varphi_{ij}\colon C_i \to C_j$ and $\varphi_{jk}\colon C_j \to C_k^{\mathrm{p}}$, e.g.



Singular TQFTs - part 1

Cook up a category of pre-foams $p\mathcal{F}$ from **S** and the identification. Now we want a singular TQFT functor $\mathcal{T}\colon p\mathcal{F} \to \mathbb{C}\text{-}\mathbf{Vect}$. In order to do so we use the two TQFTs (Frobenius algebras) $\mathcal{A}_1$ and $\mathcal{A}_2$ and decompose:



Here f_i is a surface associated to $\mathcal{A}_i$ for $i = 1, 2$.

Singular TQFTs - part 2

Define the gluing procedure:

$$\mathrm{glue}_{\mathcal{A}_1}\colon \mathcal{A}_1 \otimes \mathcal{A}_1 \to \mathcal{A}_1,\ (a{+}bX)\otimes(c+dX) \mapsto (a+bX)(c-dX),$$
$$\mathrm{glue}_{\mathcal{A}_2}\colon \mathcal{A}_2 \to \mathcal{A}_1,\ 1 \mapsto 1.$$

Then we set

$$\mathcal{T}(f_c) = (\mathrm{tr}_1)^{\otimes m}(\mathrm{glue}_{\mathcal{A}_1}^{\otimes m}(\mathcal{T}_{\mathcal{A}_1}(f_1)) \otimes \mathrm{glue}_{\mathcal{A}_2}^{\otimes m}(\mathcal{T}_{\mathcal{A}_2}(f_2))) \in \mathbb{C}^{\otimes m} \cong \mathbb{C}.$$

This gives a well-defined functor on closed pre-foams assigning to each such f_c a value $\mathcal{T}(f_c) \in \mathbb{C}$. A crucial insight of Blanchet is that this can be extended:

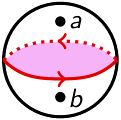
Theorem(Blanchet 2010)

This construction can be extended to a singular TQFT functor $\mathcal{T}\colon p\mathcal{F} \to \mathbb{C}\text{-}\mathbf{Vect}$.

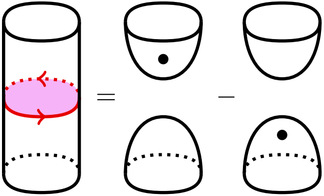
From TQFTs to $\mathbb{C}$-linear cobordism categories

Let $\mathfrak{F}$ be the $\mathbb{C}$-linear category whose objects are webs and:

- The hom spaces $\mathrm{Hom}_{\mathcal{C}}(\mathrm{web},\mathrm{web})$ is the $\mathbb{C}$-vector whose basis are all (embedded) pre-foams between these webs modulo relations.
- The relations are isotopies and some (local) relations, e.g.:



$$= \begin{cases} 1, & \text{if } a=1, b=0, \\ -1, & \text{if } a=0, b=1, \\ 0, & \text{otherwise,} \end{cases}$$



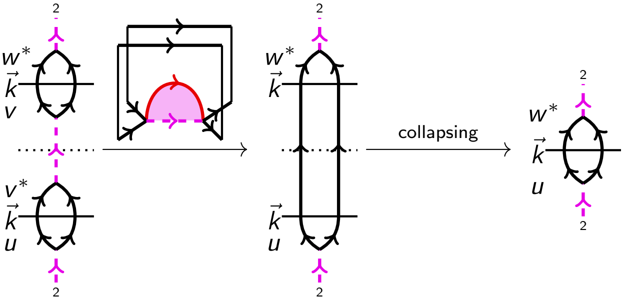
Remark

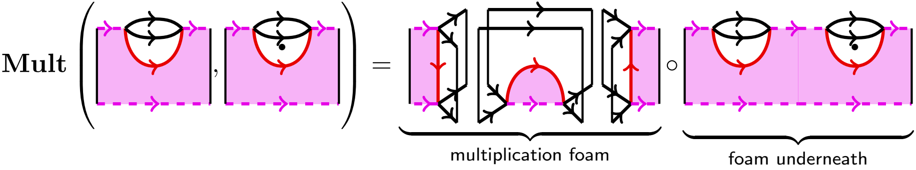
To find the relations is the most difficult part of the game.

Remark

The grading is still the (slightly rearranged) topological Euler characteristic.

Define a 'foamy" algebra $\mathbf{W}_{\vec{k}}$ as before. An example of the multiplication is





A "singular cobordism algebra"

Some nice features of the singular cobordism construction:

- One can use more general gluing maps, Frobenius algebras, work over $\mathbb{Z}$ etc.
- The signs in Khovanov homology are automatically fixed (similarly for the $\mathfrak{sl}_M/\mathfrak{gl}_M$ "friends" of Khovanov homology).
- This generalizes to $\mathfrak{sl}_M/\mathfrak{gl}_M$. This should generalize to other types as well.
- In particular, one obtains a categorification of $\mathbb{C}_q^M \otimes \cdots \otimes \mathbb{C}_q^M$ and possibly all its summands as well.
- This should give a generators/relations presentation of the M-block parabolic category $\mathcal{O}$ for $\mathfrak{gl}_m$.
- Explicit relations to algebraic geometry (e.g. Grassmannians, Springer varieties etc.) need to be worked out.
- Explicit relations to quantum Chern-Simons theory and string theory need to be worked out.
- More...

There is still much to do...

Thanks for your attention!