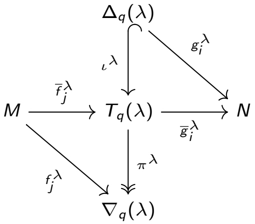
Cellular structures using $\mathbf{U}_q$-tilting modules

Or: centralizer algebras are fun!

Daniel Tubbenhauer



Joint work with Henning Haahr Andersen and Catharina Stroppel

July 2015

The main theorem

Theorem

Let T be a $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$-tilting module. Then $\mathrm{End}_{\mathbf{U}_q}(T)$ is a cellular algebra.

Thus, properties of $\mathrm{End}_{\mathbf{U}_q}(T)$ follow via roots and weight system combinatorics.

I have to explain the words in red. But let us start with an example.

Example(Schur 1901)

Let S_d be the symmetric group in d letters and let $\Delta_1(\omega_1)$ be the vector representation of $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{gl}_n)$. Take $T = \Delta_1(\omega_1)^{\otimes d}$, then

$$\Phi_{\mathrm{SW}} \colon \mathbb{K}[S_d] \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\mathrm{SW}} \colon \mathbb{K}[S_d] \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } n \geq d.$$

Since T is a $\mathbf{U}_1$-tilting module, $\mathbb{K}[S_d]$ is cellular.

1. $\mathbf{U}_q$-tilting modules
   - $\mathbf{U}_q$ and its representation theory
   - The category of $\mathbf{U}_q$-tilting modules

2. Cellularity of $\mathrm{End}_{\mathbf{U}_q}(T)$
   - Cellular algebras
   - Cellularity and $\mathbf{U}_q$-tilting modules

3. The representation theory of $\mathrm{End}_{\mathbf{U}_q}(T)$
   - Consequences of cellularity - $\mathbf{U}_q$-tilting view
   - Examples that fit into the picture

Why quantum groups and tilting modules?

- Interesting tensor categories.
- Applications in topology: link invariants, 3-manifold invariants and modular categories (Witten, Reshetikhin-Turaev, ...).
- Connections with affine Kac-Moody algebras (Kazhdan-Lusztig, ...).
- Connections with the (modular) representation theory of the symmetric group and of Ariki-Koike algebras (Lascoux-Leclerc-Thibon, ...).
- Nice combinatorics à la Kazhdan-Lusztig (Soergel, ...).
- Fusion (Andersen-Stroppel, Kazhdan-Lusztig, ...).
- Quantum cohomology (Witten, Korff-Stroppel, ...).
- More...

Quantum enveloping algebras – general $\mathfrak{g}$ works similar

Recall that $\mathfrak{sl}_2$ is generated by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $\mathbb{Q}$-algebra $\mathbf{U}_1(\mathfrak{sl}_2)$ consists of words in the symbols E, F, H modulo (plus other relations)

$$EF - FE = H.$$

Its quantum cousin, the $\mathbb{Q}(v)$-algebra $\mathbf{U}_v(\mathfrak{sl}_2)$, consists of words in the symbols $E, F, K^{\pm 1}$ modulo (plus other relations)

$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}}.$$

Roughly: $K = v^H$ and $\lim_{v \to 1} \mathbf{U}_v(\mathfrak{sl}_2) = \mathbf{U}_1(\mathfrak{sl}_2)$.

Quantized counting

The quantum integers and the quantum factorials are:

$$[a] = \frac{v^a - v^{-a}}{v^1 - v^{-1}} = v^{a-1} + v^{a-3} + \cdots + v^{-a+3} + v^{-a+1} \in \mathbb{Q}(v),$$
$$[b]! = [1] \cdots [b-1][b] \in \mathbb{Q}(v).$$

Example

For "$v = 1$" the quantum numbers are $[a] = a$. Thus, in most cases, except some "exceptional" cases, $[a]$ is a quantized version of a.

The "exceptional" cases are the ones where "$v = q \in \mathbb{K} - \{0\}$" is a root of unity with q^2 of order ℓ: $[a] = 0 \in \mathbb{K}$ iff q is a root of unity with q^2 of order ℓ.

Thus, $[3] = v^2 + 1 + v^{-2} = 0$ iff "$v = q \in \mathbb{K} - \{0\}$" is a third root of unity.

Quantum groups at roots of unity

Fix an arbitrary element $q \in \mathbb{K} - \{0\}$ and set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Define

$$\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g}) = \mathbf{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{K}.$$

Here $\mathbf{U}_{\mathcal{A}} = \mathbf{U}_{\mathcal{A}}(\mathfrak{g})$ is Lusztig's $\mathcal{A}$-form: the $\mathcal{A}$-subalgebra of $\mathbf{U}_v = \mathbf{U}_v(\mathfrak{g})$ generated by $K_i^{\pm 1}$, $E_i^{(j)} = \frac{1}{[j]!} E^j$ and $F_i^{(j)} = \frac{1}{[j]!} F^j$ for $i = 1, \dots, n-1$ and $j \in \mathbb{N}$.

Example

In the $\mathfrak{sl}_2$ case, the $\mathbb{Q}(v)$-algebra $\mathbf{U}_v(\mathfrak{sl}_2)$ is generated by K, K^{-1} and E, F subject to some relations.

Let q be a complex, primitive third root of unity. $\mathbf{U}_q(\mathfrak{sl}_2)$ is generated by $K, K^{-1}, E, F, E^{(3)}$ and $F^{(3)}$ subject to some relations. Here $E^{(3)}, F^{(3)}$ are extra generators, since $E^3 = [3]! E^{(3)} = 0$ because of $[3] = 0$.

Atoms of representation categories

What are the "atoms" of the category A-**Mod** (e.g. finite-dimensional A-module)? And how to construct or at least parametrize these "atoms"?

"Objects without substructure?" (aka, simple) or "Objects without finer decomposition?" (aka, indecomposable).

A representation category A-**Mod** is semisimple iff all objects are sums of simples. For these categories the questions are usually "easy" to answer.

Beware: dividing into semisimple representation categories and non-semisimple representation categories is like dividing the world into bananas and non-bananas.

Example(Maschke 1899, Frobenius 1900, Young 1901)

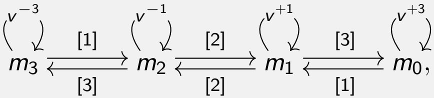
$\mathbb{K}[S_d]$-$\mathbf{Mod}_{fd}$ is semisimple iff $\mathrm{char}(\mathbb{K})$ does not divide $d!$. In this case the simple $\mathbb{K}[S_d]$-modules are parametrized by partitions aka Young diagrams.

Weyl modules as atoms

For each dominant $\mathbf{U}_v$-weight $\lambda \in X^+$ there is a simple $\mathbf{U}_v$-module $\Delta_v(\lambda)$ called Weyl module. Fact: the set $\{\Delta_v(\lambda) \mid \lambda \in X^+\}$ is a complete set of pairwise non-isomorphic, simple $\mathbf{U}_v$-modules (of type 1).

Example

For $\mathfrak{sl}_2$ we have $X^+ = \mathbb{Z}_{\geq 0}$. The Weyl module $\Delta_v(3)$ is



where E "acts to the right", F "acts to the left" and K "acts as a loop".

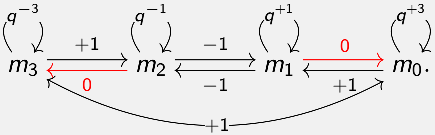
The category $\mathbf{U}_v$-$\mathbf{Mod}_{fd}$ is semisimple.

Weyl modules as atoms?

Fact: the $\Delta_q(\lambda)$'s are no longer (semi-)simple in general. But they have unique simple heads $L_q(\lambda)$. Fact: the set $\{L_q(\lambda) \mid \lambda \in X^+\}$ is a complete set of pairwise non-isomorphic, simple $\mathbf{U}_q$-modules (of type 1).

Example

Let $\mathfrak{g} = \mathfrak{sl}_2$ and q be a complex, primitive third root of unity. $\Delta_q(3)$ is



The $\mathbb{C}$-span of $\{m_1, m_2\}$ is now stable under the action of $\mathbf{U}_q(\mathfrak{sl}_2)$: this is $L_q(1)$. The simple head is $L_q(3) \cong \Delta_q(3)/L_q(1)$ and is spanned by $\{m_0, m_3\}$.

The category $\mathbf{U}_q$-$\mathbf{Mod}_{fd}$ is not semisimple in general.

$\mathbf{U}_q$-tilting modules as atoms?

Let $\Delta_q(\lambda)$ be a Weyl module and $\nabla_q(\lambda)$ its dual.
A $\mathbf{U}_q$-tilting module T is a $\mathbf{U}_q$-module with a Δ_q-filtration and a ∇_q-filtration:

$$T = M_0 \supset M_1 \supset \cdots \supset M_{k'} \supset \cdots \supset M_{k-1} \supset M_k = 0,$$
$$0 = N_0 \subset N_1 \subset \cdots \subset N_{k'} \subset \cdots \subset N_{k-1} \subset N_k = T,$$

such that $M_{k'}/M_{k'+1}$ is some $\Delta_q(\lambda)$ and $N_{k'+1}/N_{k'}$ is some $\nabla_q(\lambda)$.

Example

All $\mathbf{U}_v$-modules are $\mathbf{U}_v$-tilting modules.

For our favorite example $q^3 = 1 \in \mathbb{C}$ and $\mathfrak{g} = \mathfrak{sl}_2$: $\Delta_q(i)$ is a $\mathbf{U}_q$-tilting module iff $i = 0, 1$ or $i \equiv -1 \mod 3$.

$\mathbf{U}_q$-tilting modules as atoms.

The category of $\mathbf{U}_q$-tilting modules $\mathcal{T}$ has some nice properties:

- $\mathcal{T}$ is closed under finite direct sums and tensor products.
- The indecomposables $T_q(\lambda)$ of $\mathcal{T}$ are parametrized by $\lambda \in X^+$. They have λ as their maximal weight and contain $\Delta_q(\lambda)$ with multiplicity 1. We have

$$\Delta_q(\lambda) \xhookrightarrow{\iota^\lambda} T_q(\lambda) \xrightarrow{\pi^\lambda} \nabla_q(\lambda).$$

Example

The vector representation $\Delta_q(1)$ is a $\mathbf{U}_q(\mathfrak{sl}_2)$-tilting module. Thus, $T = \Delta_q(1)^{\otimes d}$ is. Then $T_q(d)$ is the indecomposable summand of T containing $\Delta_q(d)$.

Example

$\Delta_q(\lambda)$ is a $\mathbf{U}_q$-tilting module for minuscule λ. Thus, tensor products of these are.

The Ext-vanishing

We have for all $\lambda, \mu \in X^+$ that

$$\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \begin{cases} \mathbb{K}c^\lambda, & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0, & \text{else,} \end{cases}$$

where $c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda)$ is the $\mathbf{U}_q$-homomorphisms that sends head to socle.

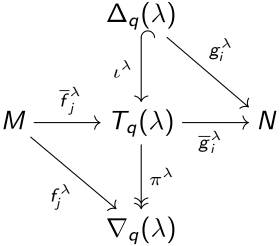
Assume that M has a Δ_q-filtration and N has a ∇_q-filtration.

- We have $\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda))$.
- We have $\dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)) = (N : \nabla_q(\lambda))$.

$\mathbf{U}_q$-tilting modules as atoms!

$$T \in \boldsymbol{\mathcal{T}} \quad \text{iff} \quad \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) \quad \text{for all } \lambda \in X^+.$$

In particular, if M has a Δ_q- and N has a ∇_q-filtration:



In words: any $\mathbf{U}_q$-homomorphism $g\colon \Delta_q(\lambda) \to N$ extends to an $\mathbf{U}_q$-homomorphism $\overline{g}\colon T_q(\lambda) \to N$ whereas any $\mathbf{U}_q$-homomorphism $f\colon M \to \nabla_q(\lambda)$ factors through $T_q(\lambda)$ via $\overline{f}\colon M \to T_q(\lambda)$.

Exempli gratia

Consequence of the discussion before:

$$\dim(\mathrm{End}_{\mathbf{U}_q}(T)) = \sum_{\lambda \in X^+} (T : \Delta_q(\lambda))^2 = \sum_{\lambda \in X^+} (T : \nabla_q(\lambda))^2.$$

Take $T = \Delta_q(\lambda)^{\otimes d}$. If $\lambda \in X^+$ is minuscule as a $\mathbf{U}_q$-weight, then $\Delta_q(\lambda)$ is always $\mathbf{U}_q$-tilting and $\dim(\mathrm{End}_{\mathbf{U}_q}(T))$ is independent of $\mathbb{K}$ and q, since $\Delta_q(\lambda)$ has a character independent of $\mathbb{K}$ and of q.

Example (Schur 1901, de Concini-Procesi 1976)

By Schur-Weyl, we see that

$$\Phi_{\mathrm{SW}} : \mathbb{K}[S_d] \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\mathrm{SW}} : \mathbb{K}[S_d] \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } n \geq d.$$

Thus, $\dim(\mathbb{K}[S_d])$ independent of $\mathbb{K}$ and q.

Exempli gratia (Temperley-Lieb without diagrams)

Let us consider our favorite case again. From the construction of $T_q(3)$:

$$\Delta_q(3) \hookrightarrow T_q(3) \twoheadrightarrow \Delta_q(1).$$

We compute:

$$T_v = \Delta_v(1) \otimes \Delta_v(1) \otimes \Delta_v(1) \cong \Delta_v(3) \oplus \Delta_v(1) \oplus \Delta_v(1),$$

whereas

$$T_q = \Delta_q(1) \otimes \Delta_q(1) \otimes \Delta_q(1) \cong T_q(3) \oplus T_q(1).$$

In particular, $\dim(\mathrm{End}_{\mathbf{U}_v(\mathfrak{sl}_2)}(T_v)) = \dim(\mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(T_q)) = 1^2 + 2^2 = 5$.

Note that $\mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(\Delta_q(1)^{\otimes d})$ is the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$ introduced by Rumer-Teller-Weyl (1932).

Cellular algebras

Definition(Graham-Lehrer 1996)

A $\mathbb{K}$-algebra A is cellular if it has a basis

$$\{c_{ij}^{\lambda} \mid \lambda \in \mathcal{P},\ i,j \in \mathcal{I}\},$$

where $(\mathcal{P}, \leq)$ is a finite poset and $\mathcal{I}^{\lambda}$ is a finite set, such that

1. The map $\mathrm{i}\colon A \to A, c_{ij}^{\lambda} \mapsto c_{ji}^{\lambda}$ is an anti-isomorphism.
2. We have (for friends of higher order)

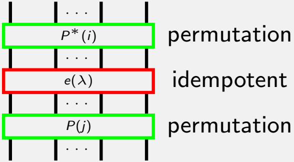
$$ac_{ij}^{\lambda} = \sum_{k \in \mathcal{I}^{\lambda}} r_{ik}(a) c_{kj}^{\lambda} + \text{friends}.$$

Note that the scalars $r_{ik}(a)$ do not depend on j. Thus, we think of the basis elements as having "independent bottom and top parts".

Prototype of a cellular basis

Example(Specht 1935, Murphy 1995)

$\mathcal{P}$ = Young diagrams λ, $\mathcal{I}^\lambda$ = standard tableaux i, j.

$c_{ij}^{\lambda} =$ 

Form $S^\lambda = \{c_j^\lambda\}$ with formal c_j^λ and action given by the $r_{ik}(a)$. The set

$$\{D^\lambda = S^\lambda/\mathrm{Rad}(S^\lambda) \mid \lambda \in \mathcal{P}_0\}$$

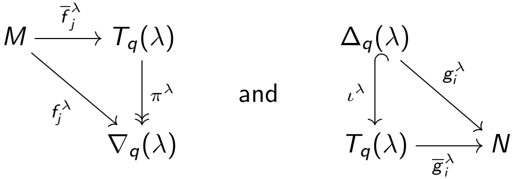
forms a complete set of pairwise non-isomorphic, simple $\mathbb{K}[S_d]$-modules.

Theorem(Graham-Lehrer 1996)

This works in general for cellular algebras.

And for $\mathrm{End}_{\mathbf{U}_q}(T)$?

Let M have a Δ_q- and N have ∇_q-filtration. Consider $\mathcal{I}^\lambda = \{1, \dots, (N : \nabla_q(\lambda))\}$ and $\mathcal{J}^\lambda = \{1, \dots, (M : \Delta_q(\lambda))\}$. By Ext-vanishing, we have diagrams



Take any bases $F^\lambda = \{f_j^\lambda \colon M \to \nabla_q(\lambda) \mid j \in \mathcal{J}^\lambda\}$ of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $G^\lambda = \{g_i^\lambda \colon \Delta_q(\lambda) \to N \mid i \in \mathcal{I}^\lambda\}$ of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$. Set

$$c_{ij}^\lambda = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(M, N)$$

for each $\lambda \in X^+$ and all $i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda$.

$\mathrm{End}_{\mathbf{U}_q}(T)$ is prototypical cellular

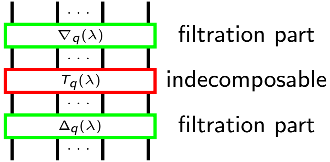
Cell datum:

- $(\mathcal{P}, \leq) = (\{\lambda \in X^+ \mid (T : \nabla_q(\lambda)) = (T : \Delta_q(\lambda)) \neq 0\}, \leq_X)$.
- $\mathcal{I}^\lambda = \{1, \dots, (T : \nabla_q(\lambda))\} = \{1, \dots, (T : \Delta_q(\lambda))\} = \mathcal{J}^\lambda$ for each $\lambda \in \mathcal{P}$.
- $\mathbb{K}$-linear anti-involution $\mathrm{i}\colon \mathrm{End}_{\mathbf{U}_q}(T) \to \mathrm{End}_{\mathbf{U}_q}(T), \phi \mapsto \mathcal{D}(\phi)$.
- Note that $\mathcal{D}(\Delta_q(\lambda)) \cong \nabla_q(\lambda)$ and $\mathcal{D}(\nabla_q(\lambda)) \cong \Delta_q(\lambda)$.
- Cellular basis $\{c_{ij}^\lambda \mid \lambda \in \mathcal{P},\ i, j \in \mathcal{I}^\lambda\}$.

Theorem

This gives a cellular datum on $\mathrm{End}_{\mathbf{U}_q}(T)$ for any $\mathbf{U}_q$-tilting module T.

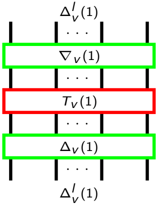
$c_{ij}^\lambda =$

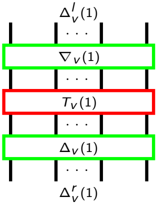


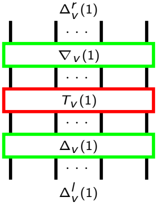
Exempli gratia (generic Temperley-Lieb)

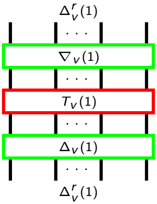
Take $\mathbb{K} = \mathbb{C}$ and $T = \Delta_v(1)^{\otimes 3} \cong \Delta_v(3) \oplus \Delta_v^l(1) \oplus \Delta_v^r(1)$. Then $\mathcal{P} = \{1, 3\}$.

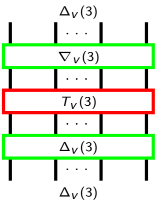
We have $\mathcal{I}^1 = \{1, 2\}$ and $\mathcal{I}^3 = \{1\}$. Thus, we have a basis

$c^1_{11} =$ 

$c^1_{12} =$ 

$c^1_{21} =$ 

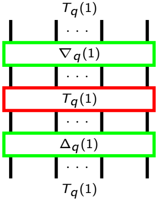
$c^1_{22} =$ 

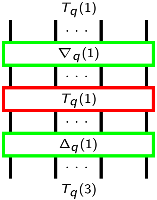
$c^3_{11} =$ 

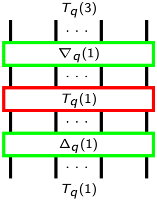
Exempli gratia (roots of unity Temperley-Lieb)

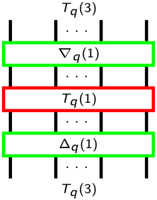
Take $T = \Delta_q(1)^{\otimes 3} \cong T_q(3) \oplus T_q(1)$. Then $\mathcal{P} = \{1, 3\}$.

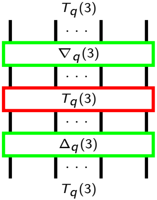
We have $\mathcal{I}^1 = \{1, 2\}$ and $\mathcal{I}^3 = \{1\}$. Consider $1 \in \mathcal{I}^1$ as indexing the factor $\Delta_q(1)$ of $T_q(1)$ and $2 \in \mathcal{I}^1$ the factor $\Delta_q(1)$ of $T_q(3)$. Thus, we have a basis

$c^1_{11} =$ 

$c^1_{12} =$ 

$c^1_{21} =$ 

$c^1_{22} =$ 

$c^3_{11} =$ 

Cellular pairing and simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules

Let T be a $\mathbf{U}_q$-tilting module. For $\lambda \in \mathcal{P}$ define ϑ^λ via

$$\mathrm{i}(h) \circ g = \vartheta^\lambda(g, h)c^\lambda, \quad g, h \in C(\lambda) = \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T).$$

Define $\mathcal{P}_0 = \{\lambda \in \mathcal{P} \mid \vartheta^\lambda \neq 0\}$ and $\mathrm{Rad}(\lambda) = \{g \in C(\lambda) \mid \vartheta^\lambda(g, C(\lambda)) = 0\}$.

Theorem(Graham-Lehrer – reinterpreted)

The set

$$\{L(\lambda) = C(\lambda)/\mathrm{Rad}(\lambda) \mid \lambda \in \mathcal{P}_0\}$$

is a complete set of pairwise non-isomorphic, simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules.

$\lambda \in \mathcal{P}_0$ iff $T_q(\lambda)$ is a summand of T. Moreover,

$$\dim(L(\lambda)) = m_\lambda, \quad T \cong \bigoplus_{\lambda \in X^+} T_q(\lambda)^{\oplus m_\lambda}.$$

Exempli gratia (Temperley-Lieb again)

Because $T_v \cong \Delta_v(3) \oplus \Delta_v(1) \oplus \Delta_v(1)$ and $T_q \cong T_q(3) \oplus T_q(1)$ we see that $\mathcal{P}_0 = \{1, 3\}$ in both cases.

In the generic case:

$$C(3) = L(3) = \{g_1^3 \colon \Delta_v(3) \to T_v\}\ ,\ C(1) = L(1) = \{g_j^1 \colon \Delta_v(1) \to T_v \mid j = 1, 2\},$$
$$\dim(L(3)) = 1 \quad \text{and} \quad \dim(L(1)) = 2.$$

In the non-semisimple case:

$$C(3) = L(3) = \{g_1^3 \colon \Delta_q(3) \to T_q\}, \qquad C(1) = \{g_j^1 \colon \Delta_q(1) \to T_q \mid j = 1, 2\},$$
$$\dim(L(3)) = 1 \quad \text{and} \quad \dim(L(1)) = 1.$$

An alternative semisimplicity criterion

Theorem(Graham-Lehrer 1996)

Let A be a cellular algebra with cell modules $C(\lambda)$ and simple modules $L(\lambda)$.

$$A \text{ is semisimple} \Leftrightarrow C(\lambda) = L(\lambda) \text{ for all } \lambda \in \mathcal{P}_0.$$

We can prove an alternative statement in our framework.

Theorem

The algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple iff T is a semisimple $\mathbf{U}_q$-module.

Corollary

The algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple iff T has only simple Weyl factors. Check this e.g. via Jantzen's sum formula.

Exempli gratia (Temperley-Lieb yet again)

Because $T_v \cong \Delta_v(3) \oplus \Delta_v(1) \oplus \Delta_v(1)$, and $\Delta_v(3)$ and $\Delta_v(1)$ are simple Weyl factors, we see that $\mathrm{End}_{\mathbf{U}_v(\mathfrak{sl}_2)}(T_v)$ is semisimple.

T_q has a Weyl factor of the form $\Delta_q(3)$. This is a non-simple Weyl factor and thus, $\mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(T_q)$ is non-semisimple.

Similarly: $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(\Delta_q(1)^{\otimes d})$ with $\delta \neq 0$ is semisimple iff q is not a root of unity in $\mathbb{K}$ or $d < \mathrm{ord}(q^2)$.

Maschke – reinterpreted

Similar as for $\mathfrak{g} = \mathfrak{sl}_2$: take $\mathfrak{g} = \mathfrak{sl}_n$ for $n \geq d$ and it follows that $\mathbb{K}[S_d] \cong \mathrm{End}_{\mathbf{U}_1(\mathfrak{sl}_n)}(\Delta_1(\omega_1)^{\otimes d})$ is semisimple iff $\mathrm{char}(\mathbb{K})$ does not divide $d!$.
Mutatis mutandis in case of the Iwahori-Hecke algebra.

A unified approach to cellularity - part 1

Note that our approach generalizes, for example to the infinite dimensional world (e.g. parabolic category $\mathcal{O}^{\mathfrak{p}}$): the following list is just the tip of the iceberg.

The following algebras fit in our set-up as well:

- The Iwahori-Hecke algebra of type **A**, by Schur-Weyl duality:

  $$\Phi_{q\mathrm{SW}}\colon \mathcal{H}_d(q) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{q\mathrm{SW}}\colon \mathcal{H}_d(q) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q}(T), \text{ if } n \geq d.$$

  This includes $\mathbb{K}[S_d]$ for $\mathrm{char}(\mathbb{K}) = p > 0$.
- $\mathfrak{sl}_2$-related algebras like Temperley-Lieb $\mathcal{TL}_d(\delta)$.
- Spider algebras $\mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_n)}(\Delta_q(\omega_{i_1}) \otimes \cdots \otimes \Delta_q(\omega_{i_d}))$.

A unified approach to cellularity - part 2

- Take $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ with $m_1 + \cdots + m_r = m$ and let V be the vector representation of $\mathbf{U}_1(\mathfrak{gl}_m)$ restricted to $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{g})$. Use $T = V^{\otimes d}$ and

  $$\Phi_{\mathrm{cl}} \colon \mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d] \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \text{ and } \Phi_{\mathrm{cl}} \colon \mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d] \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } m \geq d.$$

  This gives the cyclotomic analogon of the first point above.
- Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$. We get in the quantized case

  $$\Phi_{q\mathrm{cl}} \colon \mathcal{H}_{d,r}(q) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{q\mathrm{cl}} \colon \mathcal{H}_{d,r}(q) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q}(T), \text{ if } m \geq d,$$

  where $\mathcal{H}_{d,r}(q)$ is the Ariki-Koike algebra.
- Special cases are Iwahori-Hecke algebras of type **B**.

A unified approach to cellularity - part 3

- Let $T = \Delta_q(\omega_1)^{\otimes d}$. Let $g = \mathfrak{o}_{2n}$, $g = \mathfrak{o}_{2n+1}$ or $g = \mathfrak{sp}_{2n}$ (depending on δ).

  $$\Phi_{\mathrm{Br}}\colon \mathcal{B}_d(\delta) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\mathrm{Br}}\colon \mathcal{B}_d(\delta) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } 2n \geq d,$$

  where $\mathcal{B}_d(\delta)$ is the Brauer algebra in d strands.
- Let $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{gl}_n)$ and $T = \Delta_1(\omega_1)^{\otimes r} \otimes (\Delta_1(\omega_1)^{\otimes s})^*$:

  $$\Phi_{\mathrm{wBr}}\colon \mathcal{B}^n_{r,s}(\delta) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \text{ and } \Phi_{\mathrm{wBr}}\colon \mathcal{B}^n_{r,s}(\delta) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } n \geq r+s,$$

  where $\mathcal{B}^n_{r,s}(\delta)$ the so-called walled Brauer algebra.
- Quantizing the (walled) Brauer case: the algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is a quotient of the Birman-Murakami-Wenzl algebra $\mathcal{BMW}_d(\delta)$ and taking $n \geq 2d$ recovers $\mathcal{BMW}_d(\delta)$. Similar for the quantized walled Brauer algebra.
- Way more: quotients of these, "infinite dimensional analogons of Schur-Weyl dualities" give cyclotomic KL-R algebras etc.

There is still much to do...

Thanks for your attention!