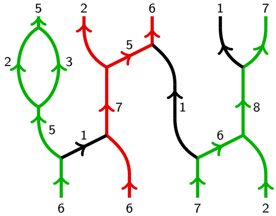
# Web calculi in representation theory

Or: the diagrammatic presentation machine

Daniel Tubbenhauer




Joint work with David Rose, Antonio Sartori, Pedro Vaz and Paul Wedrich

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# History of diagrammatic presentations in a nutshell

- **Rumer, Teller, Weyl (1932)**, Temperley-Lieb, Jones, Kauffman, Lickorish, Masbaum-Vogel, ... ($\geq$1971):
  $\mathbf{U}_q(\mathfrak{sl}_2)$-tensor category generated by $\mathbb{C}_q^2$.
- **Kuperberg (1995)**:
  $\mathbf{U}_q(\mathfrak{sl}_3)$-tensor category generated by $\Lambda_q^1\mathbb{C}_q^3 \cong \mathbb{C}_q^3$ and $\Lambda_q^2\mathbb{C}_q^3$.
- **Cautis-Kamnitzer-Morrison (2012)**:
  $\mathbf{U}_q(\mathfrak{sl}_N)$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^N$.
- **Sartori (2013)**, **Grant (2014)**:
  $\mathbf{U}_q(\mathfrak{gl}_{1|1})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{1|1}$.
- **Rose-T. (2015)**:
  $\mathbf{U}_q(\mathfrak{sl}_2)$-tensor category generated by $\mathrm{Sym}_q^k\mathbb{C}_q^2$. Thus, $\mathbf{U}_q(\mathfrak{sl}_2)$**-Mod**.
- Link polynomials: **Queffelec-Sartori (2015)**; "algebraic": **Grant (2015)**:
  $\mathbf{U}_q(\mathfrak{gl}_{N|M})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{N|M}$.
- **T.-Vaz-Wedrich (2015)**:
  $\mathbf{U}_q(\mathfrak{gl}_{N|M})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{N|M}$ and $\mathrm{Sym}_q^k\mathbb{C}_q^{N|M}$.
- **Sartori-T. (maybe! 2015)**:
  $\mathbf{U}_q(\mathfrak{so}_{2N+1}, \mathfrak{sp}_{2N}, \mathfrak{so}_{2N})$-tensor categories generated by $\Lambda_q^k\mathbb{C}_q^{2N(+1)}$.

1. Some of the first diagrammatic algebras
   - Classical Schur-Weyl duality
   - Graphical calculus via Temperley-Lieb diagrams
   - The diagrammatic presentation machine

2. The whole story for $\mathfrak{sl}_2$
   - Symmetric $\mathfrak{sl}_2$-webs
   - Proof? Symmetric Howe duality!
   - Some cousins

3. Applications
   - Link invariants à la Reshetikhin-Turaev
   - Colored Jones and HOMFLY-PT polynomials

Promise: no more $q$'s till the very end. But you can insert them everywhere.

# The question we want to solve

The symmetric group $S_m$ in $m$ letters is:

$$S_m \text{ is the the group of automorphisms of the set } \{1, \ldots, m\},$$

$$S_m = \langle \sigma_1, \ldots, \sigma_{m-1} \mid R \rangle, R = \begin{cases} \sigma_i^2 = 1, & i = 1, \ldots, m-1 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, & |i-j| = 1. \\ \sigma_i \sigma_j = \sigma_j \sigma_i, & |i-j| > 1. \end{cases}$$

The first description is given "by nature" and explains why $S_m$ is interesting. The second is a theorem and a "working horse".

Given a Lie algebra $\mathfrak{g}$, we can ask the same:

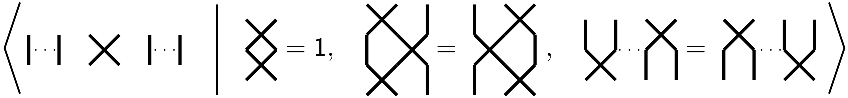
$$\mathfrak{g}\text{-}\mathbf{Mod} \rightsquigarrow \text{category of finite-dimensional } \mathbf{U}(\mathfrak{g})\text{-modules},$$

$$\mathfrak{g}\text{-}\mathbf{Mod} = \langle \mathbf{?} \mid \mathbf{??} \rangle.$$

The first description is given "by nature" and explains why $\mathfrak{g}$-**Mod** is interesting. So, we want the second as well!

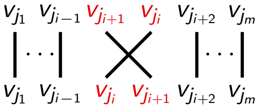
# The symmetric group - diagrammatically

The symmetric group $S_m$ can be described as:

$S_m =$ 

Similarly for $\mathbb{C}[S_m]$.

Let $\mathbb{C}^n$ with basis $v_1, \dots, v_n$. Then $\mathbb{C}[S_m]$ acts on $(\mathbb{C}^n)^{\otimes m}$ by permuting entries:



$: (\mathbb{C}^n)^{\otimes m} \to (\mathbb{C}^n)^{\otimes m}.$

This is a well-defined action (check relations!).

# The algebra $\mathbf{U}(\mathfrak{gl}_n)$

Let $\mathbf{U}(\mathfrak{gl}_n)$ be the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n$. $\mathbf{U}(\mathfrak{gl}_n)$ is given via generators and relations:

$$\mathbf{U}(\mathfrak{gl}_n) = \langle E_i, F_i, H_j \mid i = 1, \dots, n-1;\ j = 1, \dots, n \rangle \,/\text{some relations},$$

(the relations are lifts of the relations among the matrices of $\mathfrak{gl}_n$).

## Example

Recall that $\mathfrak{gl}_2$ is generated by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The $\mathbb{C}$-algebra $\mathbf{U}(\mathfrak{gl}_2)$ consists of words in the symbols $E, F, H_1, H_2$ modulo

$$EF - FE = H_1 - H_2$$

(plus a few other relations).

# $\mathbb{C}[S_m]$ is "dual" to $\mathbf{U}(\mathfrak{gl}_n)$

Since $\mathbf{U}(\mathfrak{gl}_n)$ acts "as matrices" on $\mathbb{C}^n$, we can extend it to $(\mathbb{C}^n)^{\otimes m}$ via

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes 1, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes 1, \quad \Delta(H_i) = H_i \otimes H_i.$$

## Theorem (Schur 1901)

The actions of $\mathbb{C}[S_m]$ and $\mathbf{U}(\mathfrak{gl}_n)$ on $(\mathbb{C}^n)^{\otimes m}$ commute and they generate each other commutant. In particular, they induce an algebra homomorphism

$$\Phi^m_{\mathrm{SW}} \colon \mathbb{C}[S_m] \twoheadrightarrow \mathrm{End}_{\mathbf{U}(\mathfrak{gl}_n)}((\mathbb{C}^n)^{\otimes m}),$$

$$\Phi^m_{\mathrm{SW}} \colon \mathbb{C}[S_m] \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}(\mathfrak{gl}_n)}((\mathbb{C}^n)^{\otimes m}), \text{ if } n \geq m,$$

(and of course a "dual version" which we do not need).

In words: Schur almost gave a diagrammatic generators and relations description of the full subcategory $\mathfrak{gl}_2$-**Mod**$_e$ of $\mathfrak{gl}_n$-**Mod** tensor generated by the vector representation $\mathbb{C}^n$ of $\mathbf{U}(\mathfrak{gl}_n)$.

# The 2-web space

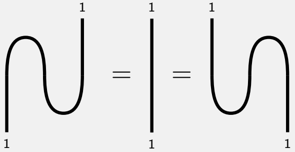
## Definition(Rumer-Teller-Weyl 1932)

The 2-*web space* $\mathrm{Hom}_{\mathbf{2\text{-}Web}}(b,t)$ is the free $\mathbb{C}$-vector space generated by non-intersecting arc diagrams with $b, t$ bottom/top boundary points modulo:

**Circle removal** :  $= -2.$



**Isotopy relations** :

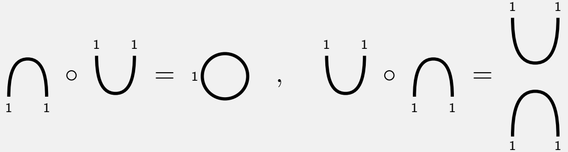


# The 2-web category

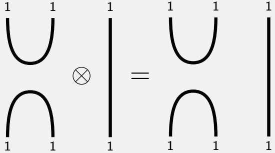
## Definition(Kuperberg 1995)

The *2-web category* **2-Web** is the (braided) monoidal, $\mathbb{C}$-linear category with:

- Objects are vectors $\vec{k} = (1, \ldots, 1)$ and morphisms are $\mathrm{Hom}_{\mathbf{2\text{-}Web}}(\vec{k}, \vec{l})$.
- Composition $\circ$:




- Tensoring $\otimes$:



# Diagrams for intertwiners

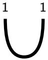
Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{sl}_2)$-intertwiners

$$\mathrm{cap}\colon \mathbb{C}^2 \otimes \mathbb{C}^2 \twoheadrightarrow \mathbb{C}, \quad \mathrm{cup}\colon \mathbb{C} \hookrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2,$$

projecting $\mathbb{C}^2 \otimes \mathbb{C}^2$ onto $\mathbb{C}$ respectively embedding $\mathbb{C}$ into $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Let $\mathfrak{sl}_2$-$\mathbf{Mod}_e$ be the (braided) monoidal, $\mathbb{C}$-linear category whose objects are tensor generated by $\mathbb{C}^2$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_e$:

$$\vec{k} = (1, \ldots, 1) \mapsto \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2,$$

$\mapsto \mathrm{cap}$ ,


$\mapsto \mathrm{cup}$

## Theorem(Folklore, Rumer-Teller-Weyl 1932)

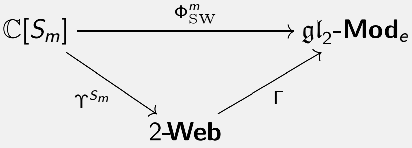
$\Gamma\colon 2\text{-}\mathbf{Web}^{\oplus} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_e$ is an equivalence of (braided) monoidal categories.

# The diagrammatic presentation machine

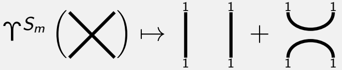
Consider $\mathbb{C}[S_m]$ as a $\mathbb{C}$-linear category. By Schur-Weyl duality there is a full functor $\Phi^m_{\mathrm{SW}} \colon \mathbb{C}[S_m] \to \mathfrak{gl}_2\text{-}\mathbf{Mod}_e$.

## Theorem

Define 2-**Web** such there is a commutative diagram




with




$\Upsilon^{S_m} \rightsquigarrow$ circle relation, isotopy relations, $\quad \ker(\Phi^m_{\mathrm{SW}}) \rightsquigarrow$ isotopy relations

# From $\mathfrak{gl}_2$ to $\mathfrak{sl}_2$

Restricting from $\mathfrak{gl}_2$ to $\mathfrak{sl}_2$ could increase the number of intertwiners:

$$\mathbf{U}(\mathfrak{sl}_2) \subset \mathbf{U}(\mathfrak{gl}_2) \quad \Rightarrow \quad \mathrm{Hom}_{\mathbf{U}(\mathfrak{sl}_2)}(M, M') \supset \mathrm{Hom}_{\mathbf{U}(\mathfrak{gl}_2)}(M, M').$$

Note that $\mathbb{C}^2$ is self-dual as a $\mathbf{U}(\mathfrak{sl}_2)$-module, but not as a $\mathbf{U}(\mathfrak{gl}_2)$-module. We obtain extra diagrams:

$: \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C},$ 

$: \mathbb{C} \to \mathbb{C}^2 \otimes \mathbb{C}^2.$

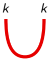
These satisfy the isotopy relations and "fill up the missing" hom-spaces:

$\mathrm{Hom}_{\mathbf{U}(\mathfrak{gl}_2)}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}) = 0,$ but $\mathrm{Hom}_{\mathbf{U}(\mathfrak{sl}_2)}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}) = \Big\langle$ 

$\Big\rangle$, *etc.*

A red $\mathfrak{sl}_2$-*web* is a labeled trivalent graph locally generated by

$\mathrm{cap}_k =$ 

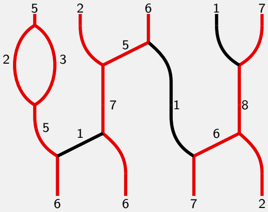
, $\mathrm{cup}^k =$ 

, $\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{s}_{k+l}^{k,l} =$ 


Here $k, l, k + l \in \{0, 1, \ldots\}$.

## Example



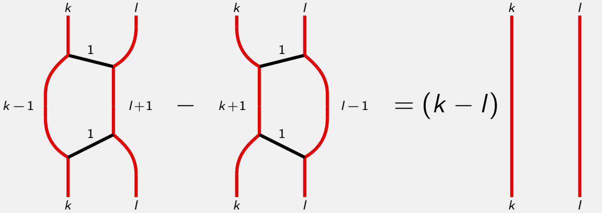
# Let us form a category again

Define the (braided) monoidal, $\mathbb{C}$-linear category $2\text{-}\mathbf{Web}_{\mathrm{r}}$ by using:

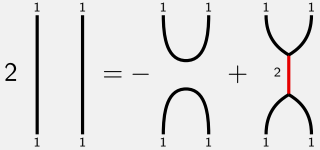
## Definition

The red *2-web space* $\mathrm{Hom}_{2\text{-}\mathbf{Web}_{\mathrm{r}}}(\vec{k},\vec{l})$ is the free $\mathbb{C}$-vector space generated by red 2-webs modulo the circle removal, isotopies and:

$\mathfrak{gl}_m$ "**ladder**" **relations**, e.g. :




**Dumbbell relation** :



# Diagrams for intertwiners

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{sl}_2)$-intertwiners

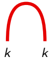
$$\mathrm{cap}_k\colon \mathrm{Sym}^k\mathbb{C}^2 \otimes \mathrm{Sym}^k\mathbb{C}^2 \twoheadrightarrow \mathbb{C}, \quad \mathrm{cup}^k\colon \mathbb{C} \hookrightarrow \mathrm{Sym}^k\mathbb{C}^2 \otimes \mathrm{Sym}^k\mathbb{C}^2,$$

$$\mathrm{m}_{k,l}^{k+l}\colon \mathrm{Sym}^k\mathbb{C}^2 \otimes \mathrm{Sym}^l\mathbb{C}^2 \twoheadrightarrow \mathrm{Sym}^{k+l}\mathbb{C}^2, \quad \mathrm{s}_{k+l}^{k,l}\colon \mathrm{Sym}^{k+l}\mathbb{C}^2 \hookrightarrow \mathrm{Sym}^k\mathbb{C}^2 \otimes \mathrm{Sym}^l\mathbb{C}^2$$

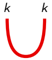
given by projection and inclusion.

Let $\mathfrak{sl}_2$-$\mathbf{Mod}_s$ be the (braided) monoidal, $\mathbb{C}$-linear category whose objects are tensor generated by $\mathrm{Sym}^k\mathbb{C}^2$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{r}} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_s$:

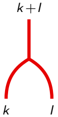
$$\vec{k} = (k_1, \dots, k_m) \mapsto \mathrm{Sym}^{k_1}\mathbb{C}^2 \otimes \cdots \otimes \mathrm{Sym}^{k_m}\mathbb{C}^2,$$




$\mapsto \mathrm{cap}_k$ ,




$\mapsto \mathrm{cup}^k$ ,




$\mapsto \mathrm{m}_{k,l}^{k+l}$ ,


$\mapsto \mathrm{s}_{k+l}^{k,l}$

## Theorem

$\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{r}}^{\oplus} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_s$ is an equivalence of (braided) monoidal categories.

# “Howe” to prove this?

Howe: the commuting actions of $\mathbf{U}(\mathfrak{gl}_m)$ and $\mathbf{U}(\mathfrak{gl}_N)$ on

$$\mathrm{Sym}^K(\mathbb{C}^m \otimes \mathbb{C}^N) \cong \bigoplus_{k_1+\cdots+k_m=K} (\mathrm{Sym}^{k_1}\mathbb{C}^N \otimes \cdots \otimes \mathrm{Sym}^{k_m}\mathbb{C}^N)$$

introduce an $\mathbf{U}(\mathfrak{gl}_m)$-action $f$ on the right term with $\vec{k}$-weight space $\mathrm{Sym}^{\vec{k}}\mathbb{C}^N$.

In particular, there is a functorial action

$$\Phi^m_{\mathrm{sym}} \colon \dot{\mathbf{U}}(\mathfrak{gl}_m) \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_s,$$

$$\vec{k} \mapsto \mathrm{Sym}^{\vec{k}}\mathbb{C}^N, \quad X \in 1_{\vec{l}}\mathbf{U}(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{gl}_N\text{-}\mathbf{Mod}_s}(\mathrm{Sym}^{\vec{k}}\mathbb{C}^N, \mathrm{Sym}^{\vec{l}}\mathbb{C}^N).$$

Howe: $\Phi^m_{\mathrm{sym}}$ is full. Or in words:

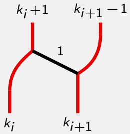
relations in $\dot{\mathbf{U}}(\mathfrak{gl}_m)$ + kernel of $\Phi^m_{\mathrm{sym}} \rightsquigarrow$ relations in $\mathfrak{gl}_N\text{-}\mathbf{Mod}_s$.

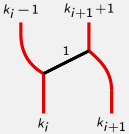
## Theorem

Define 2-**Web**$_{\mathrm{r}}$ such there is a commutative diagram

$$\dot{\mathbf{U}}(\mathfrak{gl}_m) \xrightarrow{\Phi^m_{\mathrm{sym}}} \mathfrak{gl}_2\text{-}\mathbf{Mod}_s, \qquad \dot{\mathbf{U}}(\mathfrak{gl}_m) \xrightarrow{\Upsilon^m} 2\text{-}\mathbf{Web}_{\mathrm{r}} \xrightarrow{\Gamma} \mathfrak{gl}_2\text{-}\mathbf{Mod}_s$$

with

$\Upsilon^m(E_i 1_{\vec{k}}) \mapsto$ 

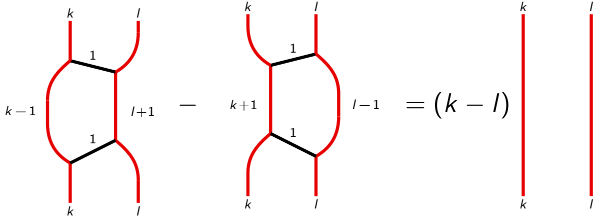
, $\Upsilon^m(F_i 1_{\vec{k}}) \mapsto$ 


$\Upsilon^m \rightsquigarrow \mathfrak{gl}_m$ "ladder" relations, $\ker(\Phi^m_{\mathrm{sym}}) \rightsquigarrow$ dumbbell relation.

# Exempli gratia

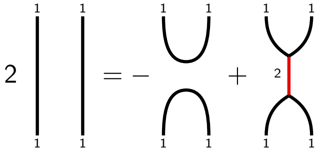
The $\mathfrak{gl}_m$ "ladder" relations come up as follows:

$$EF\mathbf{1}_{\vec{\mathbf{k}}} - FE\mathbf{1}_{\vec{\mathbf{k}}} = (k-l)\mathbf{1}_{\vec{\mathbf{k}}} \rightsquigarrow$$




The dumbbell relation comes up as follows:

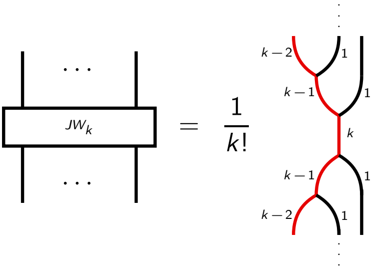
$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \bigwedge^2 \mathbb{C}^2 \oplus \mathrm{Sym}^2 \mathbb{C}^2 \cong \mathbb{C} \oplus \mathrm{Sym}^2 \mathbb{C}^2 \rightsquigarrow$$



# No fancy stuff like Karoubi completions needed

Fact: all irreducible $\mathbf{U}(\mathfrak{sl}_2)$-modules are of the form $\mathrm{Sym}^k\mathbb{C}^2$ for some $k$. Thus, $\mathfrak{sl}_2$-$\mathbf{Mod}_s$ contains all finite-dimensional representations.

In particular, the Jones-Wenzl projectors of the TL algebra (RTW algebra)

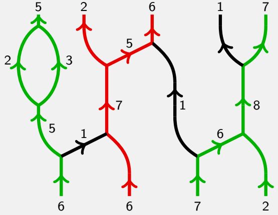



are encoded (and also all their relations!).

# As far as we can go in type $\mathrm{A}$

We could also consider $\mathfrak{sl}_N$ instead of $\mathfrak{sl}_2$ (diagram category $N$-$\mathbf{Web}_{\mathrm{r}}$). And $\bigwedge^k\mathbb{C}^N$ instead of $\mathrm{Sym}^k\mathbb{C}^N$ (diagram category $N$-$\mathbf{Web}_{\mathrm{g}}$). Or both together (diagram category $N$-$\mathbf{Web}_{\mathrm{gr}}$). The graphical calculi for these are very similar.

## Example




$$\text{green } k \leftrightsquigarrow \textstyle\bigwedge^k\mathbb{C}^N,$$

$$\text{red } k \leftrightsquigarrow \mathrm{Sym}^k\mathbb{C}^N,$$

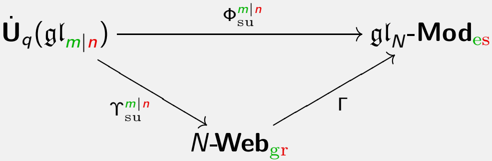
$$\text{black } 1 \leftrightsquigarrow \textstyle\bigwedge^1\mathbb{C}^N \cong \mathrm{Sym}^1\mathbb{C}^N \cong \mathbb{C}^N.$$

# The machine in action again

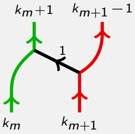
They are look the same because they are spit out by our machine, e.g.:

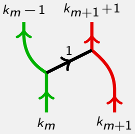
## Theorem

Define $N$-$\mathbf{Web}_{\mathrm{gr}}$ such there is a commutative diagram




with

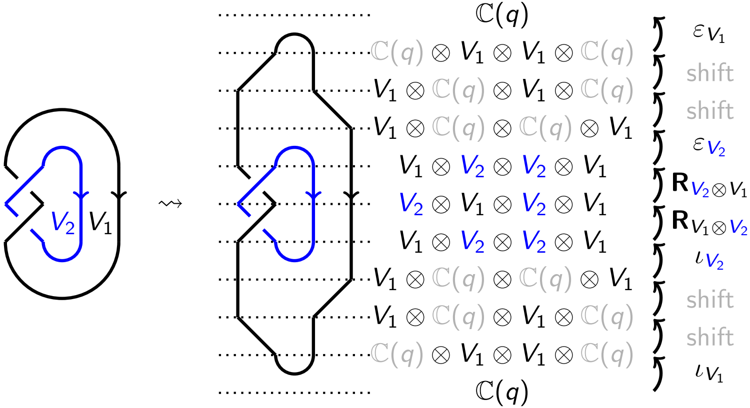
$\Upsilon^{m|n}_{\mathrm{su}}(E_m 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^{m|n}_{\mathrm{su}}(F_m 1_{\vec{k}}) \mapsto$ 


$\Upsilon^{m|n}_{\mathrm{su}} \rightsquigarrow$ "$\mathfrak{gl}_{m|n}$ ladder" relations, $\ker(\Phi^{m|n}_{\mathrm{su}}) \rightsquigarrow$ the exterior relation.

# Link invariants via representation theory

Color link components with $\mathbf{U}_q(\mathfrak{g})$-modules. Put the links into a Morse position.

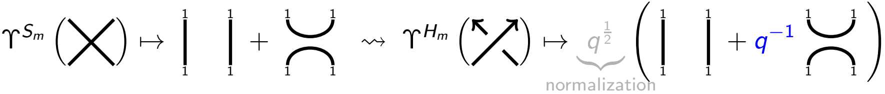



## Theorem (Reshetikhin-Turaev 1990)

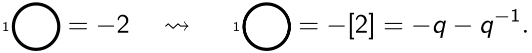
The composite $\mathcal{P}^q_{\vec{V}}(1) \in \mathbb{Q}(q)$ is an invariant of (framed, oriented) links.

# Wait: we have a diagrammatic calculus

Recall that there was an action of $\mathbb{C}[S_m]$ on 2-**Web**. This quantizes:




Similarly, our diagrammatic calculus quantizes. The difference is

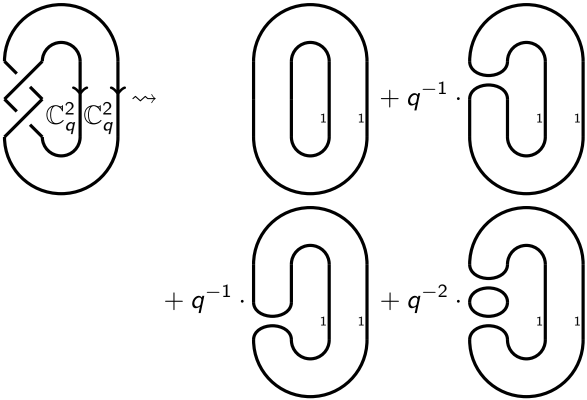



## Theorem (Kauffman 1987)

Using these in the Reshetikhin-Turaev set-up with $\mathfrak{g} = \mathfrak{sl}_2$ and only $\mathbb{C}_q^2$ as colors gives a combinatorial way to compute the Jones polynomial.

There is a framing shift which I hide, but never mind.

# Exempli gratia

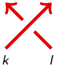



$$\rightsquigarrow \mathcal{P}^q_{\mathbb{C}_q^2,\mathbb{C}_q^2}(\mathcal{L}) = q((-q-q^{-1})^2 + q^{-1}(-q-q^{-1}) + q^{-1}(-q-q^{-1}) + q^{-2}(-q-q^{-1})^2) = q(q+q^{-1})(q+q^{-3}).$$

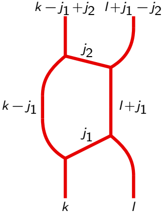
This is (up to normalization) the Jones polynomial of the Hopf link.

# Wait: we have even more diagrammatic calculi

We can quantize the category 2-**Web**$_{\mathrm{r}}$ and obtain a braided monoidal category which enables us to cook up link invariants diagrammatically. The braiding is:




$$= \underbrace{(-1)^k q^{-k-\frac{kl}{2}}}_{\text{normalization}} \sum_{\substack{j_1,j_2\geq 0 \\ j_1-j_2=k-l}} (-q)^{j_1}$$




## Theorem

Using these in the Reshetikhin-Turaev set-up with $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathrm{Sym}^k_q\mathbb{C}^2_q$ as colors gives a new, combinatorial way to compute the colored Jones polynomial.

This works completely similar for the categories $N$-**Web**$_{\mathrm{g}}$, $N$-**Web**$_{\mathrm{r}}$ and $N$-**Web**$_{\mathrm{gr}}$ giving rise to a new way to compute colored $\mathfrak{sl}_N$ polynomials for all colors (and thus, colored HOMFLY-PT polynomials).

# Another application: the HOMFLY-PT symmetry

There is also a polynomial called colored HOMFLY-PT polynomial $\mathcal{P}_{\lambda}^{a,q}(\mathcal{K}) \in \mathbb{C}(a,q)$ ($\mathcal{K}$"="knot). The colors $\lambda$ are Young diagrams. The whole framework should be seen as the "$N \to \infty$"-version of the $\mathfrak{sl}_N$ Reshetikhin-Turaev approach ($a \rightsquigarrow q^N$) with $\lambda$ corresponding to irreducible highest weight module.

From the diagrammatic calculi we obtain:

**Corollary (the HOMFLY-PT symmetry)**

The colored HOMFLY-PT polynomial satisfies

$$\mathcal{P}_{\lambda}^{a,q}(\mathcal{K}) = (-1)^{co}\mathcal{P}_{\lambda^{\mathrm{T}}}^{a,q^{-1}}(\mathcal{K}),$$

where $co$ is some constant. Similar for links.

This is a representation theoretical explanation of the the HOMFLY-PT symmetry.

# I do not have tenure. So I have to bore you a bit more.

Some additional remarks.

- Homework: feed the machine with your favorite duality.
- We are working on the type $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$-versions and the diagrams work fine (yet, the quantization is complicated).
- Some parts even work in the non-semisimple case (e.g. at roots of unities).
- The whole approach seems to be amenable to categorification.
- Relations to categorifications of the Hecke algebra using Soergel bimodules or category $\mathcal{O}$ need to be worked out.
- This could lead to a categorification of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{m|n})$ (since the "complicated" super relations are build in the calculus).
- A "green-red-foamy" approach could shed additional light on colored Khovanov-Rozansky homologies.

There is still much to do...

Thanks for your attention!