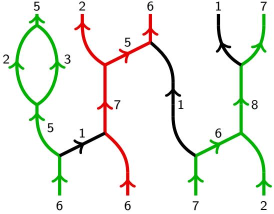
Web calculi in representation theory

Or: the diagrammatic presentation machine

Daniel Tubbenhauer



Joint work with David Rose, Antonio Sartori, Pedro Vaz and Paul Wedrich

August 2015

History of diagrammatic presentations in a nutshell

- **Rumer, Teller, Weyl (1932)**, Temperley-Lieb, Jones, Kauffman, Lickorish, Masbaum-Vogel, ... ($\geq$1971):
  $\mathbf{U}_q(\mathfrak{sl}_2)$-tensor category generated by $\mathbb{C}_q^2$.
- **Kuperberg (1995)**:
  $\mathbf{U}_q(\mathfrak{sl}_3)$-tensor category generated by $\Lambda_q^1\mathbb{C}_q^3 \cong \mathbb{C}_q^3$ and $\Lambda_q^2\mathbb{C}_q^3$.
- **Cautis-Kamnitzer-Morrison (2012)**:
  $\mathbf{U}_q(\mathfrak{sl}_N)$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^N$.
- **Sartori (2013)**, **Grant (2014)**:
  $\mathbf{U}_q(\mathfrak{gl}_{1|1})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{1|1}$.
- **Rose-T. (2015)**:
  $\mathbf{U}_q(\mathfrak{sl}_2)$-tensor category generated by $\mathrm{Sym}_q^k\mathbb{C}_q^2$. Thus, $\mathbf{U}_q(\mathfrak{sl}_2)$**-Mod**.
- Link polynomials: **Queffelec-Sartori (2015)**; "algebraic": **Grant (2015)**:
  $\mathbf{U}_q(\mathfrak{gl}_{N|M})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{N|M}$.
- **T.-Vaz-Wedrich (2015)**:
  $\mathbf{U}_q(\mathfrak{gl}_{N|M})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{N|M}$ and $\mathrm{Sym}_q^k\mathbb{C}_q^{N|M}$.
- **Sartori-T. (maybe! 2015)**:
  $\mathbf{U}_q(\mathfrak{so}_{2N+1}, \mathfrak{sp}_{2N}, \mathfrak{so}_{2N})$-tensor categories generated by $\Lambda_q^k\mathbb{C}_q^{2N(+1)}$.

1. The story for $\mathfrak{sl}_2$
   - Graphical calculus via Temperley-Lieb diagrams
   - The full story for $\mathfrak{sl}_2$
   - Proof? Symmetric Howe duality!
2. Exterior $\mathfrak{gl}_N$-web categories
   - Its cousins: the N-webs
   - Proof? Skew Howe duality!
3. As far as we can go in type $\mathbf{A}_{N-1}$
   - Even more cousins: the green-red N-webs
   - Proof? Super Howe duality!
4. The machine in action – yet again
   - What happens in types $\mathbf{B}_N$, $\mathbf{C}_N$ and $\mathbf{D}_N$?
   - This!

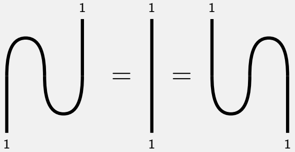
Promise: no more q's from now on. But you can insert them everywhere if you like.

The 2-web space

Definition(Rumer-Teller-Weyl 1932)

The 2-*web space* $\mathrm{Hom}_{2\text{-}\mathbf{Web}}(b, t)$ is the free $\mathbb{C}$-vector space generated by non-intersecting arc diagrams with b, t bottom/top boundary points modulo:

Circle removal :  $= -2.$

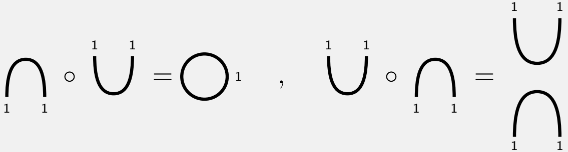
Isotopy relations : 

The 2-web category

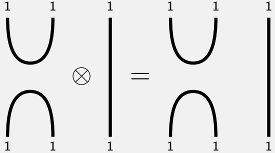
Definition(Kuperberg 1995)

The 2-*web category* 2-**Web** is the (braided) monoidal, $\mathbb{C}$-linear category with:

- Objects are vectors $\vec{k} = (1, \ldots, 1)$ and morphisms are $\mathrm{Hom}_{2\text{-}\mathbf{Web}}(\vec{k}, \vec{l})$.
- Composition $\circ$:



- Tensoring $\otimes$:



Diagrams for intertwiners

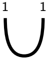
Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{sl}_2)$-intertwiners

$$\mathrm{cap}\colon \mathbb{C}^2 \otimes \mathbb{C}^2 \twoheadrightarrow \mathbb{C}, \quad \mathrm{cup}\colon \mathbb{C} \hookrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2,$$

projecting $\mathbb{C}^2 \otimes \mathbb{C}^2$ onto $\mathbb{C}$ respectively embedding $\mathbb{C}$ into $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Let $\mathfrak{sl}_2$-**Mod** be the (braided) monoidal, $\mathbb{C}$-linear category whose objects are tensor generated by $\mathbb{C}^2$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}$:

$$\vec{k} = (1, \ldots, 1) \mapsto \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2,$$

$\mapsto \mathrm{cap}$,


$\mapsto \mathrm{cup}$

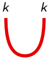
Theorem(Folklore)

$\Gamma\colon 2\text{-}\mathbf{Web}^{\oplus} \to \mathfrak{sl}_2\text{-}\mathbf{Mod}$ is an equivalence of (braided) monoidal categories.

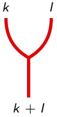
The symmetric story

A red $\mathfrak{sl}_2$-*web* is a labeled trivalent graph locally made of

$\mathrm{cap}_k =$ 

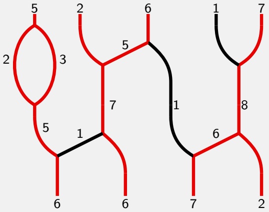
, $\mathrm{cup}^k =$ 

, $\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{s}_{k+l}^{k,l} =$ 

Here $k, l, k+l \in \{0, 1, \ldots\}$.

Example



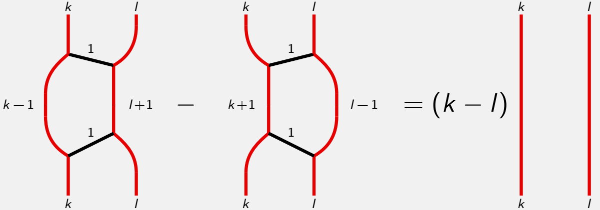
Let us form a category again

Define the (braided) monoidal, $\mathbb{C}$-linear category 2-$\mathbf{Web}_{\mathrm{r}}$ by using:

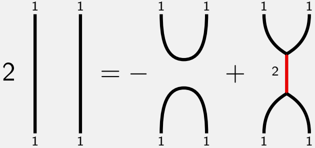
Definition

The red 2-*web space* $\mathrm{Hom}_{2\text{-}\mathbf{Web}_{\mathrm{r}}}(\vec{k},\vec{l})$ is the free $\mathbb{C}$-vector space generated by red 2-webs modulo the circle removal, isotopies and:

$\mathfrak{gl}_m$ **"ladder" relations** :



Dumbbell relation :



Diagrams for intertwiners

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{sl}_2)$-intertwiners

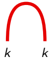
$$\mathrm{cap}_k\colon \mathrm{Sym}^k\mathbb{C}^2\otimes\mathrm{Sym}^k\mathbb{C}^2\twoheadrightarrow\mathbb{C},\quad \mathrm{cup}^k\colon\mathbb{C}\hookrightarrow\mathrm{Sym}^k\mathbb{C}^2\otimes\mathrm{Sym}^k\mathbb{C}^2,$$

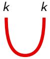
$$\mathrm{m}_{k,l}^{k+l}\colon\mathrm{Sym}^k\mathbb{C}^2\otimes\mathrm{Sym}^l\mathbb{C}^2\twoheadrightarrow\mathrm{Sym}^{k+l}\mathbb{C}^2,\quad \mathrm{s}_{k+l}^{k,l}\colon\mathrm{Sym}^{k+l}\mathbb{C}^2\hookrightarrow\mathrm{Sym}^k\mathbb{C}^2\otimes\mathrm{Sym}^l\mathbb{C}^2$$

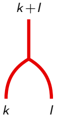
given by projection and inclusion.

Let $\mathfrak{sl}_2$-$\mathbf{Mod}_s$ be the (braided) monoidal, $\mathbb{C}$-linear category whose objects are tensor generated by $\mathrm{Sym}^k\mathbb{C}^2$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{r}}\to\mathfrak{sl}_2\text{-}\mathbf{Mod}_s$:

$$\vec{k}=(k_1,\dots,k_m)\mapsto\mathrm{Sym}^{k_1}\mathbb{C}^2\otimes\cdots\otimes\mathrm{Sym}^{k_m}\mathbb{C}^2,$$



$\mapsto \mathrm{cap}_k$,


$\mapsto \mathrm{cup}^k$,


$\mapsto \mathrm{m}_{k,l}^{k+l}$,


$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem

$\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{r}}^{\oplus}\to\mathfrak{sl}_2\text{-}\mathbf{Mod}_s$ is an equivalence of (braided) monoidal categories.

“Howe” to prove this?

Howe: the commuting actions of $\mathbf{U}(\mathfrak{gl}_m)$ and $\mathbf{U}(\mathfrak{gl}_N)$ on

$$\mathrm{Sym}^K(\mathbb{C}^m \otimes \mathbb{C}^N) \cong \bigoplus_{k_1+\dots+k_m=K} (\mathrm{Sym}^{k_1}\mathbb{C}^N \otimes \dots \otimes \mathrm{Sym}^{k_m}\mathbb{C}^N)$$

introduce an $\mathbf{U}(\mathfrak{gl}_m)$-action f on the right term with $\vec{k}$-weight space $\mathrm{Sym}^{\vec{k}}\mathbb{C}^N$.

In particular, there is a functorial action

$$\Phi^m_{\mathrm{sym}} \colon \dot{\mathbf{U}}(\mathfrak{gl}_m) \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_s,$$

$$\vec{k} \mapsto \mathrm{Sym}^{\vec{k}}\mathbb{C}^N, \quad X \in 1_{\vec{l}}\mathbf{U}(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{gl}_N\text{-}\mathbf{Mod}_s}(\mathrm{Sym}^{\vec{k}}\mathbb{C}^N, \mathrm{Sym}^{\vec{l}}\mathbb{C}^N).$$

Howe: Φ^m_{sym} is full. Or in words:

relations in $\dot{\mathbf{U}}(\mathfrak{gl}_m)$ + kernel of $\Phi^m_{\mathrm{sym}} \rightsquigarrow$ relations in $\mathfrak{gl}_N\text{-}\mathbf{Mod}_s$.

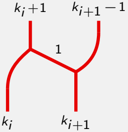
The diagrammatic presentation machine

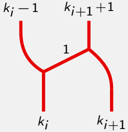
Theorem

Define $2\text{-}\mathbf{Web}_{\mathrm{r}}$ such there is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}(\mathfrak{gl}_m) & \xrightarrow{\Phi^m_{\mathrm{sym}}} & \mathfrak{gl}_2\text{-}\mathbf{Mod}_s \\ & \searrow^{\Upsilon^m} \quad \nearrow_{\Gamma} & \\ & 2\text{-}\mathbf{Web}_{\mathrm{r}} & \end{array}$$

with

$\Upsilon^m(E_i 1_{\vec{k}}) \mapsto$ 

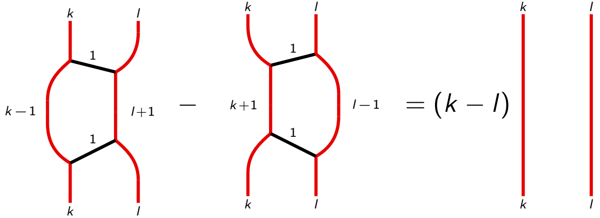
, $\Upsilon^m(F_i 1_{\vec{k}}) \mapsto$ 

$\Upsilon^m \rightsquigarrow \mathfrak{gl}_m$ "ladder" relations, $\ker(\Phi^m_{\mathrm{sym}}) \rightsquigarrow$ dumbbell relation.

Exempli gratia

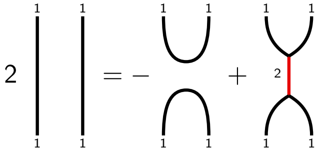
The $\mathfrak{gl}_{m}$ "ladder" relations come up as follows:

$$EF\mathbf{1}_{\vec{\mathbf{k}}} - FE\mathbf{1}_{\vec{\mathbf{k}}} = (k-l)\mathbf{1}_{\vec{\mathbf{k}}} \rightsquigarrow$$



The dumbbell relation comes up as follows:

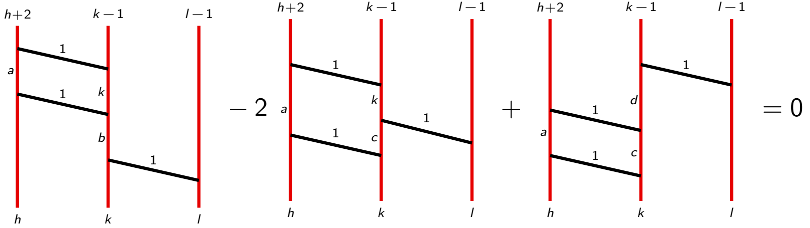
$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \bigwedge^{2}\mathbb{C}^2 \oplus \mathrm{Sym}^{2}\mathbb{C}^2 \cong \mathbb{C} \oplus \mathrm{Sym}^{2}\mathbb{C}^2 \rightsquigarrow$$



It is even better than expected!

The hardest $\mathfrak{gl}_m$ "ladder" relations, e.g. Serre relations as

$$E_i^2 E_{i+1} \mathbf{1}_{\vec{\mathbf{k}}} - 2 E_i E_{i+1} E_i \mathbf{1}_{\vec{\mathbf{k}}} + E_{i+1} E_i^2 \mathbf{1}_{\vec{\mathbf{k}}} = 0 \rightsquigarrow$$

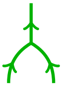


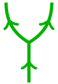
do not have to be forced to hold, but are consequences. This pattern repeats in for other web categories.

Morally: web categories have a very economic presentation!

Replace red by green and add orientations

A green *N-web* is an oriented, labeled, trivalent graph locally made of

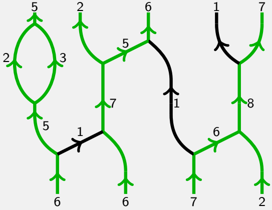
$\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{s}_{k+l}^{k,l} =$ 

$k, l, k+l \in \mathbb{N}$

(and some caps, cups and signs that I skip today).

Example



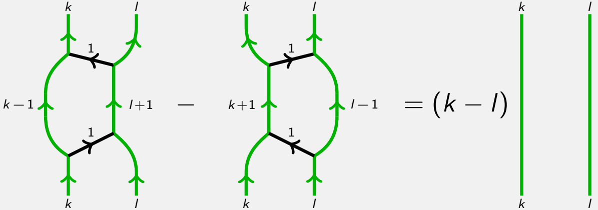
Let us form a category again

Define the (braided) monoidal, $\mathbb{C}$-linear category N-$\mathbf{Web}_{\mathrm{g}}$ by using:

Definition(Cautis-Kamnitzer-Morrison 2012)

The green *N-web space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathrm{g}}}(\vec{k},\vec{l})$ is the free $\mathbb{C}$-vector space generated by green N-webs modulo isotopies and:

$\mathfrak{gl}_m$ **"ladder" relations** :



Exterior relation :

$= 0$, if $k > N$.

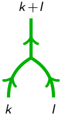
Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_N)$-intertwiners

$$\mathrm{m}_{k,l}^{k+l}\colon \textstyle\bigwedge^k \mathbb{C}^N \otimes \bigwedge^l \mathbb{C}^N \twoheadrightarrow \bigwedge^{k+l}\mathbb{C}^N \quad , \quad \mathrm{s}_{k+l}^{k,l}\colon \bigwedge^{k+l}\mathbb{C}^N \hookrightarrow \bigwedge^k \mathbb{C}^N \otimes \bigwedge^l \mathbb{C}^N$$

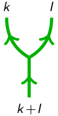
given by projection and inclusion.

Let $\mathfrak{gl}_N$-$\mathbf{Mod}_e$ be the (braided) monoidal, $\mathbb{C}$-linear category whose objects are tensor generated by $\bigwedge^k \mathbb{C}^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e$:

$$\vec{k} = (k_1, \dots, k_m) \mapsto \textstyle\bigwedge^{k_1}\mathbb{C}^N \otimes \cdots \otimes \bigwedge^{k_m}\mathbb{C}^N,$$



$\mapsto \mathrm{m}_{k,l}^{k+l}$,



$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem(Cautis-Kamnitzer-Morrison 2012)

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}}^{\oplus} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e$ is an equivalence of (braided) monoidal categories.

“Howe” to prove this?

Howe: the commuting actions of $\mathbf{U}(\mathfrak{gl}_m)$ and $\mathbf{U}(\mathfrak{gl}_N)$ on

$$\Lambda^K(\mathbb{C}^m \otimes \mathbb{C}^N) \cong \bigoplus_{k_1+\cdots+k_m=K} (\Lambda^{k_1}\mathbb{C}^N \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^N)$$

introduce an $\mathbf{U}(\mathfrak{gl}_m)$-action f on the right term with $\vec{k}$-weight space $\Lambda^{\vec{k}}\mathbb{C}^N$.

In particular, there is a functorial action

$$\Phi^m_{\mathrm{skew}} \colon \dot{\mathbf{U}}(\mathfrak{gl}_m) \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e,$$

$$\vec{k} \mapsto \Lambda_q^{\vec{k}}\mathbb{C}^N, \quad X \in 1_{\vec{l}}\mathbf{U}(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{gl}_N\text{-}\mathbf{Mod}_e}(\Lambda_q^{\vec{k}}\mathbb{C}^N, \Lambda_q^{\vec{l}}\mathbb{C}^N).$$

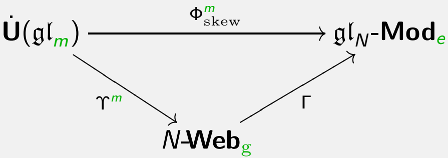
Howe: Φ^m_{skew} is full. Or in words:

relations in $\dot{\mathbf{U}}(\mathfrak{gl}_m)$ + kernel of Φ^m_{skew} $\rightsquigarrow$ relations in $\mathfrak{gl}_N\text{-}\mathbf{Mod}_e$.

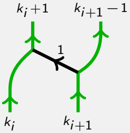
Define the diagrams to make this work

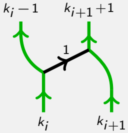
Theorem(Cautis-Kamnitzer-Morrison 2012)

Define N-**Web**$_{\mathrm{g}}$ such there is a commutative diagram



with

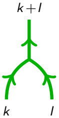
$\Upsilon^m(E_i 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^m(F_i 1_{\vec{k}}) \mapsto$ 

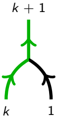
$\Upsilon^m \rightsquigarrow \mathfrak{gl}_m$ "ladder" relations, $\ker(\Phi^m_{\text{skew}}) \rightsquigarrow$ exterior relation.

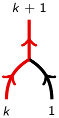
Could there be a pattern?

A green-red *N-web* is a colored, labeled, trivalent graph locally made of

$\mathrm{m}^{k+l}_{k,l} =$ 

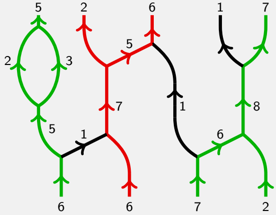
, $\mathrm{m}^{k+l}_{k,l} =$ 

, $\mathrm{m}^{k+l}_{k,1} =$ 

, $\mathrm{m}^{k+l}_{k,1} =$ 

And of course splits and some mirrors as well!

Example



The green-red N-web category

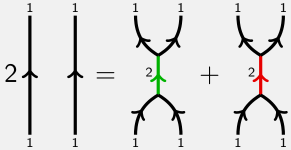
Define the (braided) monoidal, $\mathbb{C}$-linear category N-$\mathbf{Web}_{\mathrm{gr}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}, \vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$. The green-red *$N$-web space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathrm{gr}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}$-vector space generated by N-webs modulo isotopies and:

$\mathfrak{gl}_m + \mathfrak{gl}_n$ **"ladder" relations** : same as before, but now in green and red!

Dumbbell relation :



Exterior relation :

$= 0$, if $k > N$.

Diagrams for intertwiners - Part 4

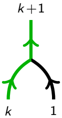
Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_N)$-intertwiners

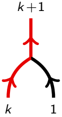
$$\mathrm{m}_{k,1}^{k+1}\colon \textstyle\bigwedge^k\mathbb{C}^N \otimes \mathbb{C}^N \twoheadrightarrow \textstyle\bigwedge^{k+1}\mathbb{C}^N, \quad \mathrm{m}_{k,1}^{k+1}\colon \mathrm{Sym}^k\mathbb{C}^N \otimes \mathbb{C}^N \twoheadrightarrow \mathrm{Sym}^{k+1}\mathbb{C}^N$$

plus others as before.

Let $\mathfrak{gl}_N$-$\mathbf{Mod}_{\mathrm{es}}$ be the (braided) monoidal, $\mathbb{C}$-linear category whose objects are tensor generated by $\bigwedge^k\mathbb{C}^N, \mathrm{Sym}^k\mathbb{C}^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$:

$$\vec{k} = (k_1, \dots, k_m, k_{m+1}, \dots, k_{m+n}) \mapsto \textstyle\bigwedge^{k_1}\mathbb{C}^N \otimes \cdots \otimes \mathrm{Sym}^{k_{m+n}}\mathbb{C}^N,$$



$\mapsto \mathrm{m}_{k,1}^{k+1}$,


$\mapsto \mathrm{m}_{k,1}^{k+1}$, $\cdots$

Theorem

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}}^{\oplus} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$ is an equivalence of (braided) monoidal categories.

Super $\mathfrak{gl}_{m|n}$

Definition

The *general linear superalgebra* $\mathbf{U}(\mathfrak{gl}_{m|n})$ is generated by H_i and F_i, E_i subject the some relations, most notably, the *super relations*:

$$E_m^2 = 0 = F_m^2, \qquad H_m + H_{m+1} = F_m E_m + E_m F_m,$$
$$2E_m E_{m+1} E_{m-1} E_m = E_m E_{m+1} E_m E_{m-1} + E_{m-1} E_m E_{m+1} E_m$$
$$+ E_{m+1} E_m E_{m-1} E_m + E_m E_{m-1} E_m E_{m+1} \text{ (plus an F version).}$$

There is a Howe pair $(\mathbf{U}(\mathfrak{gl}_{m|n}), \mathbf{U}(\mathfrak{gl}_N))$ with $\vec{k} = (k_1, \dots, k_{m+n})$-weight space under the $\mathbf{U}(\mathfrak{gl}_{m|n})$-action on $\bigwedge^K(\mathbb{C}^{m|n} \otimes \mathbb{C}^N)$ given by

$$\bigwedge^{k_1}\mathbb{C}^N \otimes \cdots \bigwedge^{k_m}\mathbb{C}^N \otimes \mathrm{Sym}^{k_{m+1}}\mathbb{C}^N \otimes \cdots \otimes \mathrm{Sym}^{k_{m+n}}\mathbb{C}^N.$$

An aside: everything works for green-red $\mathbf{U}(\mathfrak{gl}_{N|M})$-webs as well, with the Howe pair $(\mathbf{U}(\mathfrak{gl}_{m|n}), \mathbf{U}(\mathfrak{gl}_{N|M}))$.

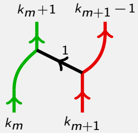
Define the diagrams to make this work - yet again

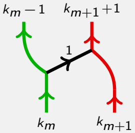
Theorem

Define N-$\mathbf{Web}_{\mathrm{gr}}$ such there is a commutative diagram

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_{m|n}) \xrightarrow{\Phi^{m|n}_{\mathrm{su}}} \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}, \quad \dot{\mathbf{U}}_q(\mathfrak{gl}_{m|n}) \xrightarrow{\Upsilon^{m|n}_{\mathrm{su}}} N\text{-}\mathbf{Web}_{\mathrm{gr}} \xrightarrow{\Gamma} \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$$

with

$\Upsilon^{m|n}_{\mathrm{su}}(E_m 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^{m|n}_{\mathrm{su}}(F_m 1_{\vec{k}}) \mapsto$ 

$\Upsilon^{m|n}_{\mathrm{su}} \rightsquigarrow$ "$\mathfrak{gl}_{m|n}$ ladder" relations, $\ker(\Phi^{m|n}_{\mathrm{su}}) \rightsquigarrow$ the exterior relation.

Another meal for our machine

Howe: the commuting actions of $\mathbf{U}(\mathfrak{so}_{2m})$ and $\mathbf{U}(\mathfrak{so}_{2N(+1)})$ on

$$\bigwedge^{K}(\mathbb{C}^{m} \otimes \mathbb{C}^{2N(+1)}) \cong \bigoplus_{k_1+\cdots+k_n=K} \bigwedge^{\vec{k}}\mathbb{C}^{2N(+1)}$$

introduce an $\mathbf{U}(\mathfrak{so}_{2m})$-action f with $\vec{k}$-weight space $\bigwedge^{\vec{k}}\mathbb{C}^{2N(+1)}$.

In particular, there is a functorial action

$$\Phi^{m}_{\mathrm{so}} \colon \dot{\mathbf{U}}(\mathfrak{so}_{2m}) \to \mathfrak{so}_{2N(+1)}\text{-}\mathbf{Mod}_{\mathrm{e}},$$
$$\vec{k} \mapsto \bigwedge^{\vec{k}}\mathbb{C}^{2N(+1)}, \quad \text{etc..}$$

Howe: Φ^{m}_{so} is full. Or in words:

relations in $\dot{\mathbf{U}}(\mathfrak{so}_{2m})$ + kernel of $\Phi^{m}_{\mathrm{so}} \rightsquigarrow$ relations in $\mathfrak{so}_{2N(+1)}$-$\mathbf{Mod}_{\mathrm{e}}$.

And another one

Howe: the commuting actions of $\mathbf{U}(\mathfrak{sp}_{2m})$ and $\mathbf{U}(\mathfrak{sp}_{2N})$ on

$$\textstyle\bigwedge^K(\mathbb{C}^m \otimes \mathbb{C}^{2N}) \cong \bigoplus_{k_1+\cdots+k_n=K} \bigwedge^{\vec{k}}\mathbb{C}^{2N}$$

introduce an $\mathbf{U}(\mathfrak{sp}_{2m})$-action f with $\vec{k}$-weight space $\bigwedge^{\vec{k}}\mathbb{C}^{2N}$.

In particular, there is a functorial action

$$\Phi^m_{\mathrm{sp}} \colon \dot{\mathbf{U}}(\mathfrak{sp}_{2m}) \to \mathfrak{sp}_{2N}\text{-}\mathbf{Mod}_{\mathrm{e}},$$
$$\vec{k} \mapsto \textstyle\bigwedge^{\vec{k}}\mathbb{C}^{2N}, \quad \text{etc.}$$

Howe: Φ^m_{sp} is full. Or in words:

relations in $\dot{\mathbf{U}}(\mathfrak{sp}_{2m})$ + kernel of $\Phi^m_{\mathrm{sp}} \rightsquigarrow$ relations in $\mathfrak{sp}_{2N}\text{-}\mathbf{Mod}_{\mathrm{e}}$.

The definition of the diagrams is already determined

Theorem

Define N-$\mathbf{BDWeb}_{\mathrm{g}}$ such there is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}(\mathfrak{so}_{2m}) & \xrightarrow{\Phi^{2m}_{\mathrm{so}}} & \mathfrak{so}_{2N(+1)}\text{-}\mathbf{Mod}_{\mathrm{e}} \\ & {\scriptstyle \Upsilon^{m}_{\mathrm{so}}} \searrow \quad \nearrow {\scriptstyle \Gamma} & \\ & N\text{-}\mathbf{BDWeb}_{\mathrm{g}} & \end{array}$$

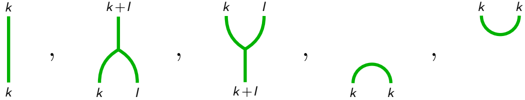
Define N-$\mathbf{CWeb}_{\mathrm{g}}$ such there is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}(\mathfrak{sp}_{2m}) & \xrightarrow{\Phi^{2m}_{\mathrm{sp}}} & \mathfrak{sp}_{2N}\text{-}\mathbf{Mod}_{\mathrm{e}} \\ & {\scriptstyle \Upsilon^{m}_{\mathrm{sp}}} \searrow \quad \nearrow {\scriptstyle \Gamma} & \\ & N\text{-}\mathbf{CWeb}_{\mathrm{g}} & \end{array}$$

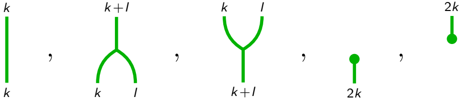
$\Upsilon^{m}_{\mathrm{so}} \rightsquigarrow \mathfrak{so}_{2m}$ "ladder" relations, $\Upsilon^{m}_{\mathrm{sp}} \rightsquigarrow \mathfrak{sp}_{2m}$ "ladder" relations etc.

Green type **BCD**-webs

Green *webs in types* $\mathbf{B}_N$ and $\mathbf{D}_N$ are generated by



Green *webs in type* $\mathbf{C}_N$ are generated by



The lanterns reflect the fact that $\bigwedge^k \mathbb{C}^{2N}$ is not irreducible in type $\mathbf{C}_N$:

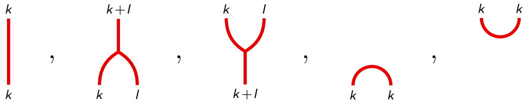
$\rightsquigarrow$ slantern: $\bigwedge^k \mathbb{C}^{2N} \twoheadrightarrow \mathbb{C}$,


$\rightsquigarrow$ plantern: $\mathbb{C} \hookrightarrow \bigwedge^k \mathbb{C}^{2N}$

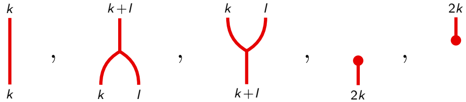
Red type **BCD**-webs

There are also Howe pairs $(\mathbf{U}(\mathfrak{sp}_{2m}),\mathbf{U}(\mathfrak{so}_{2n(+1)}))$ and $(\mathbf{U}(\mathfrak{so}_{2m}),\mathbf{U}(\mathfrak{sp}_{2n}))$ acting now on the symmetric tensors. Guess what comes out: red webs!

Red *webs in type* $\mathbf{C}_N$ are generated by



Red *webs in types* $\mathbf{B}_N$ and $\mathbf{D}_N$ are generated by



The lanterns reflect the fact that $\mathrm{Sym}^k\mathbb{C}^{2N(+1)}$ is not irreducible in types $\mathbf{B}_N,\mathbf{D}_N$:

$\rightsquigarrow$ slantern: $\mathrm{Sym}^{2k}\mathbb{C}^{2N}\twoheadrightarrow\mathbb{C}$,

$\rightsquigarrow$ plantern: $\mathbb{C}\hookrightarrow\mathrm{Sym}^{2k}\mathbb{C}^{2N}$

I do not have tenure. So I have to bore you a bit more.

Some additional remarks.

- Homework: feed the machine with your favorite duality.
- Everything quantizes without too many difficulties. The quantized version sheds new light on HOMFLY-PT, Kauffman and Reshetikhin-Turaev polynomials: their symmetries can be explained representation theoretical.
- Some parts even work in the non-semisimple case (e.g. at roots of unities).
- The whole approach seems to be amenable to categorification.
- Relations to categorifications of the Hecke algebra using Soergel bimodules or category $\mathcal{O}$ need to be worked out.
- This could lead to a categorification of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{m|n})$ (since the "complicated" super relations are build in the calculus).
- A "green-red-foamy" approach could shed additional light on colored Khovanov-Rozansky homologies.

There is still much to do...

Thanks for your attention!