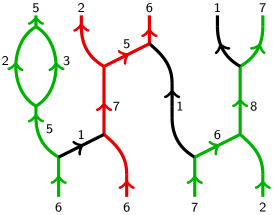
From dualities to diagrams

Or: the diagrammatic presentation machine

Daniel Tubbenhauer



Joint work with David Rose, Pedro Vaz and Paul Wedrich

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1. Exterior $\mathfrak{gl}_N$-web categories
   - Graphical calculus via Temperley-Lieb diagrams
   - Its cousins: the N-webs
   - Proof? Skew quantum Howe duality!

2. Exterior-symmetric $\mathfrak{gl}_N$-web categories
   - Even more cousins: the green-red N-webs
   - Proof? Super quantum Howe duality!

3. The machine in action – yet again
   - Super-Super duality and even more cousins
   - Braidings and applications

History of diagrammatic presentations in a nutshell

- Rumer, Teller, Weyl (1932):
  $\mathbf{U}(\mathfrak{sl}_2)$-tensor category generated by $\mathbb{C}^2$.
- Temperley-Lieb, Jones, Kauffman, Lickorish, Masbaum-Vogel ... ($\geq$1971):
  $\mathbf{U}_q(\mathfrak{sl}_2)$-tensor category generated by $\mathbb{C}_q^2$.
- Kuperberg (1995):
  $\mathbf{U}_q(\mathfrak{sl}_3)$-tensor category generated by $\Lambda_q^1\mathbb{C}_q^3 \cong \mathbb{C}_q^3$ and $\Lambda_q^2\mathbb{C}_q^3$.
- Cautis-Kamnitzer-Morrison (2012):
  $\mathbf{U}_q(\mathfrak{sl}_N)$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^N$.
- Sartori (2013), Grant (2014):
  $\mathbf{U}_q(\mathfrak{gl}_{1|1})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{1|1}$.
- Rose-T. (2015):
  $\mathbf{U}_q(\mathfrak{sl}_2)$-tensor category generated by $\mathrm{Sym}_q^k\mathbb{C}_q^2$.
- Link polynomials: Queffelec-Sartori (2015); "algebraic": Grant (2015):
  $\mathbf{U}_q(\mathfrak{gl}_{N|M})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{N|M}$.

"Howe" do they fit in one framework?

The 2-web space

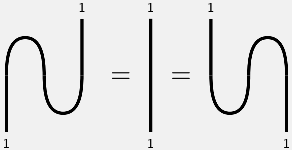
Definition(Rumer-Teller-Weyl 1932)

The 2-*web space* $\mathrm{Hom}_{2\text{-}\mathbf{Web}}(b,t)$ is the free $\mathbb{C}_q = \mathbb{C}(q)$-vector space generated by non-intersecting arc diagrams with b,t bottom/top boundary points modulo:

Circle removal :

$= -q - q^{-1} = -[2].$

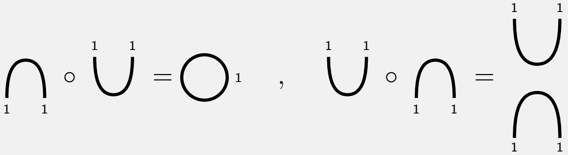
Isotopy relations :



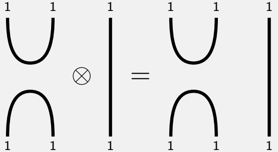
Definition(Kuperberg 1995)

The 2-*web category* 2-**Web** is the (braided) monoidal, $\mathbb{C}_q$-linear category with:

- Objects are vectors $\vec{k} = (1, \dots, 1)$ and morphisms are $\mathrm{Hom}_{2\text{-}\mathbf{Web}}(\vec{k}, \vec{l})$.
- Composition $\circ$:



- Tensoring $\otimes$:



Diagrams for intertwiners

Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_2)$-intertwiners

$$\mathrm{cap}\colon \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \twoheadrightarrow \mathbb{C}_q \quad \text{and} \quad \mathrm{cup}\colon \mathbb{C}_q \hookrightarrow \mathbb{C}_q^2 \otimes \mathbb{C}_q^2,$$

projecting $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$ onto $\mathbb{C}_q$ respectively embedding $\mathbb{C}_q$ into $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$.

Let $\mathfrak{gl}_2$-**Mod** be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\mathbb{C}_q^2$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web} \to \mathfrak{gl}_2\text{-}\mathbf{Mod}$:

$$\vec{k} = (1, \dots, 1) \mapsto \mathbb{C}_q^2 \otimes \cdots \otimes \mathbb{C}_q^2,$$

$\mapsto \mathrm{cap}$,

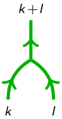

$\mapsto \mathrm{cup}$

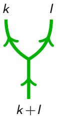
Theorem(Folklore)

$\Gamma\colon 2\text{-}\mathbf{Web}^{\oplus} \to \mathfrak{gl}_2\text{-}\mathbf{Mod}$ is an equivalence of (braided) monoidal categories.

The main step beyond $\mathfrak{gl}_2$: trivalent vertices

An *N-web* is an oriented, labeled, trivalent graph locally made of

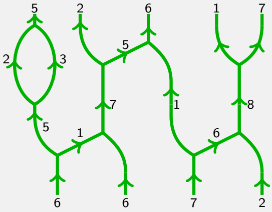
$\mathrm{m}^{k+l}_{k,l} =$ 

, $\mathrm{s}^{k,l}_{k+l} =$ 

$k, l, k+l \in \mathbb{N}$

(and no pivotal things today).

Example



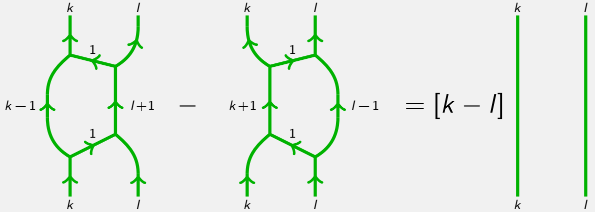
Let us form a category again

Define the (braided) monoidal, $\mathbb{C}_q$-linear category N-$\mathbf{Web}_{\mathfrak{g}}$ by using:

Definition(Cautis-Kamnitzer-Morrison 2012)

The *N-web space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathfrak{g}}}(\vec{k},\vec{l})$ is the free $\mathbb{C}_q$-vector space generated by N-webs with $\vec{k}$ and $\vec{l}$ at the bottom and top modulo isotopies and:

$\mathfrak{gl}_m$ "ladder" relations :



Exterior relation :

$= 0$, if $k > N$.

Diagrams for intertwiners

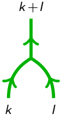
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_N)$-intertwiners

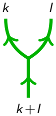
$$\mathrm{m}_{k,l}^{k+l}\colon \Lambda_q^k\mathbb{C}_q^N\otimes\Lambda_q^l\mathbb{C}_q^N\twoheadrightarrow\Lambda_q^{k+l}\mathbb{C}_q^N \quad\text{and}\quad \mathrm{s}_{k+l}^{k,l}\colon \Lambda_q^{k+l}\mathbb{C}_q^N\hookrightarrow\Lambda_q^k\mathbb{C}_q^N\otimes\Lambda_q^l\mathbb{C}_q^N$$

given by projection and inclusion.

Let $\mathfrak{gl}_N$-$\mathbf{Mod}_e$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\Lambda_q^k\mathbb{C}_q^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}}\to\mathfrak{gl}_N\text{-}\mathbf{Mod}_e$:

$$\vec{k}=(k_1,\dots,k_m)\mapsto\Lambda_q^{k_1}\mathbb{C}_q^N\otimes\cdots\otimes\Lambda_q^{k_m}\mathbb{C}_q^N,$$



$\mapsto \mathrm{m}_{k,l}^{k+l}$,


$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem(Cautis-Kamnitzer-Morrison 2012)

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}}^{\oplus}\to\mathfrak{gl}_N\text{-}\mathbf{Mod}_e$ is an equivalence of (braided) monoidal categories.

"Howe" to prove this?

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_m)$ and $\mathbf{U}_q(\mathfrak{gl}_N)$ on

$$\Lambda_q^K(\mathbb{C}_q^m \otimes \mathbb{C}_q^N) \cong \bigoplus_{k_1+\cdots+k_m=K} (\Lambda_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \Lambda_q^{k_m}\mathbb{C}_q^N)$$

introduce an $\mathbf{U}_q(\mathfrak{gl}_m)$-action f on the right term with $\vec{k}$-weight space $\Lambda_q^{\vec{k}}\mathbb{C}_q^N$.

In particular, there is a functorial action

$$\Phi^m_{\mathrm{skew}} \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e,$$

$$\vec{k} \mapsto \Lambda_q^{\vec{k}}\mathbb{C}_q^N, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{gl}_N\text{-}\mathbf{Mod}_e}(\Lambda_q^{\vec{k}}\mathbb{C}_q^N, \Lambda_q^{\vec{l}}\mathbb{C}_q^N).$$

Howe: Φ^m_{skew} is full. Or in words:

relations in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ + kernel of $\Phi^m_{\mathrm{skew}} \rightsquigarrow$ relations in $\mathfrak{gl}_N$-$\mathbf{Mod}_e$.

Define the diagrams to make this work

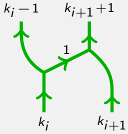
Theorem(Cautis-Kamnitzer-Morrison 2012)

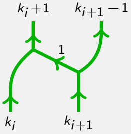
Define N-$\mathbf{Web}_{\mathrm{g}}$ such there is a commutative diagram

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_m) \xrightarrow{\Phi^m_{\mathrm{skew}}} \mathfrak{gl}_N\text{-}\mathbf{Mod}_e$$

$$\Upsilon^m : \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to N\text{-}\mathbf{Web}_{\mathrm{g}}, \qquad \Gamma : N\text{-}\mathbf{Web}_{\mathrm{g}} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e$$

with

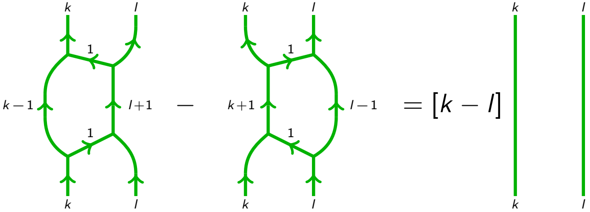
$\Upsilon^m(F_i 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^m(E_i 1_{\vec{k}}) \mapsto$ 

$\Upsilon^m \rightsquigarrow$ "$\mathfrak{gl}_m$ ladder" relations , $\ker(\Phi^m_{\mathrm{skew}}) \rightsquigarrow$ exterior relation.

Exempli gratia

The "$\mathfrak{gl}_m$ ladder" relation



is just

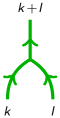
$$EF1_{\vec{k}} - FE1_{\vec{k}} = [k-l]1_{\vec{k}}.$$

The exterior relation is a diagrammatic version of

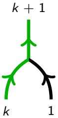
$$\Lambda_q^{>N}\mathbb{C}_q^N \cong 0.$$

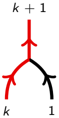
Could there be a pattern?

A green-red N-web is a colored, labeled, trivalent graph locally made of

$\mathrm{m}^{k+l}_{k,l} =$ 

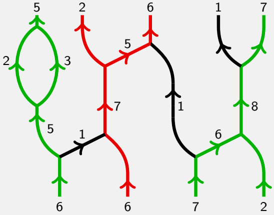
, $\mathrm{m}^{k+l}_{k,l} =$ 

, $\mathrm{m}^{k+l}_{k,1} =$ 

, $\mathrm{m}^{k+l}_{k,1} =$ 

And of course splits and some mirrors as well!

Example



The green-red N-web category

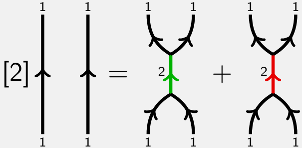
Define the (braided) monoidal, $\mathbb{C}_q$-linear category N-$\mathbf{Web}_{\mathrm{gr}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}, \vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$. The green-red *$N$-web space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathrm{gr}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by N-webs between $\vec{k}$ and $\vec{l}$ modulo isotopies and:

$\mathfrak{gl}_m + \mathfrak{gl}_n$ "ladder" relations : same as before, but now in red as well!

Dumbbell relation :



Exterior relation :

$= 0$, if $k > N$.

Diagrams for intertwiners - Part 3

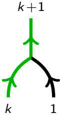
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_N)$-intertwiners

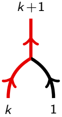
$$\mathrm{m}_{k,1}^{k+1}\colon \textstyle\bigwedge_q^k \mathbb{C}_q^N \otimes \mathbb{C}_q^N \twoheadrightarrow \bigwedge_q^{k+1}\mathbb{C}_q^N \quad \text{and} \quad \mathrm{m}_{k,1}^{k+1}\colon \mathrm{Sym}_q^k \mathbb{C}_q^N \otimes \mathbb{C}_q^N \twoheadrightarrow \mathrm{Sym}_q^{k+1}\mathbb{C}_q^N$$

plus others as before.

Let $\mathfrak{gl}_N$-$\mathbf{Mod}_{\mathrm{es}}$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\bigwedge_q^k \mathbb{C}_q^N, \mathrm{Sym}_q^k\mathbb{C}_q^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$:

$$\vec{k} = (k_1, \dots, k_m, k_{m+1}, \dots, k_{m+n}) \mapsto \textstyle\bigwedge_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_{m+n}}\mathbb{C}_q^N,$$



$\mapsto \mathrm{m}_{k,1}^{k+1}$,


$\mapsto \mathrm{m}_{k,1}^{k+1}$, $\cdots$

Theorem

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}}^{\oplus} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$ is an equivalence of (braided) monoidal categories.

Super $\mathfrak{gl}_{m|n}$

Definition

The *quantum general linear superalgebra* $\mathbf{U}_q(\mathfrak{gl}_{m|n})$ is generated by $L_i^{\pm 1}$ and F_i, E_i subject the some relations, most notably, the *super relations*:

$$F_m^2 = 0 = E_m^2 \quad , \quad \frac{L_m L_{m+1}^{-1} - L_m^{-1} L_{m+1}}{q - q^{-1}} = F_m E_m + E_m F_m,$$

$$\begin{aligned}[2] F_m F_{m+1} F_{m-1} F_m = & F_m F_{m+1} F_m F_{m-1} + F_{m-1} F_m F_{m+1} F_m \\ & + F_{m+1} F_m F_{m-1} F_m + F_m F_{m-1} F_m F_{m+1} \text{ (plus an E version).}\end{aligned}$$

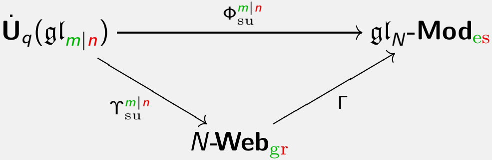
There is a Howe pair $(\mathbf{U}_q(\mathfrak{gl}_{m|n}), \mathbf{U}_q(\mathfrak{gl}_N))$ with $\vec{k} = (k_1, \ldots, k_{m+n})$-weight space under the $\mathbf{U}_q(\mathfrak{gl}_{m|n})$-action on $\Lambda_q^K(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^N)$ given by

$$\Lambda_q^{k_1}\mathbb{C}_q^N \otimes \cdots \Lambda_q^{k_m}\mathbb{C}_q^N \otimes \mathrm{Sym}_q^{k_{m+1}}\mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_{m+n}}\mathbb{C}_q^N.$$

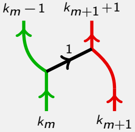
Define the diagrams to make this work

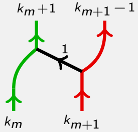
Theorem

Define N-$\mathbf{Web}_{\mathrm{gr}}$ such there is a commutative diagram



with

$\Upsilon^{m|n}_{\mathrm{su}}(F_m 1_{\vec{k}}) \mapsto$ 

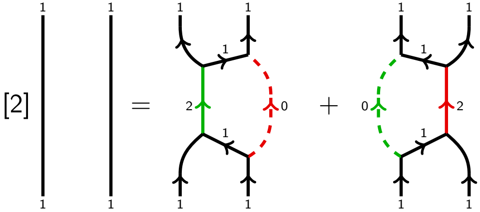
, $\Upsilon^{m|n}_{\mathrm{su}}(E_m 1_{\vec{k}}) \mapsto$ 

$\Upsilon^{m|n}_{\mathrm{su}} \rightsquigarrow$ "$\mathfrak{gl}_{m|n}$ ladder" relations , $\ker(\Phi^{m|n}_{\mathrm{su}}) \rightsquigarrow$ the exterior relation.

Exempli gratia

The dumbbell relation is the super commutator relation:

$$[2]1_{(1,1)} = F_m E_m 1_{(1,1)} + E_m F_m 1_{(1,1)}$$



$$\mathbb{C}_q^N \otimes \mathbb{C}_q^N \cong \Lambda_q^{2} \mathbb{C}_q^N \oplus \mathrm{Sym}_q^{2} \mathbb{C}_q^N.$$

All other super relations are consequences!

Another meal for our machine

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_{m|n})$ and $\mathbf{U}_q(\mathfrak{gl}_{N|M})$ on

$$\Lambda_q^K(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{N|M}) \cong \bigoplus_{k_1+\cdots+k_n=K} (\Lambda_q^{\vec{k}_0}\mathbb{C}_q^{N|M} \otimes \mathrm{Sym}_q^{\vec{k}_1}\mathbb{C}_q^{N|M})$$

introduce an $\mathbf{U}_q(\mathfrak{gl}_{m|n})$-action f with $\vec{k}$-weight space $\Lambda_q^{\vec{k}_0}\mathbb{C}_q^{N|M} \otimes \mathrm{Sym}_q^{\vec{k}_1}\mathbb{C}_q^{N|M}$.

In particular, there is a functorial action

$$\Phi_{\mathrm{susu}}^{m|n} \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_{m|n}) \to \mathfrak{gl}_{N|M}\text{-}\mathbf{Mod}_{\mathrm{es}},$$
$$\vec{k} \mapsto \Lambda_q^{\vec{k}_0}\mathbb{C}_q^{N|M} \otimes \mathrm{Sym}_q^{\vec{k}_1}\mathbb{C}_q^{N|M}, \quad \text{etc..}$$

Howe: $\Phi_{\mathrm{susu}}^{m|n}$ is full. Or in words:

relations in $\dot{\mathbf{U}}_q(\mathfrak{gl}_{m|n})$ + kernel of $\Phi_{\mathrm{susu}}^{m|n} \rightsquigarrow$ relations in $\mathfrak{gl}_{N|M}\text{-}\mathbf{Mod}_{\mathrm{es}}$.

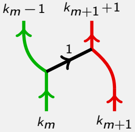
The definition of the diagrams is already determined

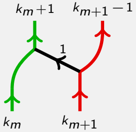
Theorem

Define $N|M\text{-}\mathbf{Web}_{\mathrm{gr}}$ such there is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_q(\mathfrak{gl}_{m|n}) & \xrightarrow{\Phi^{m|n}_{\mathrm{susu}}} & \mathfrak{gl}_{N|M}\text{-}\mathbf{Mod}_{\mathrm{es}} \\ & \searrow^{\Upsilon^{m|n}_{\mathrm{susu}}} \quad \nearrow_{\Gamma} & \\ & N|M\text{-}\mathbf{Web}_{\mathrm{gr}} & \end{array}$$

with

$\Upsilon^{m|n}_{\mathrm{susu}}(F_m 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^{m|n}_{\mathrm{susu}}(E_m 1_{\vec{k}}) \mapsto$ 

$\Upsilon^{m|n}_{\mathrm{susu}} \rightsquigarrow$ "$\mathfrak{gl}_{m|n}$ ladder" relations , $\ker(\Phi^{m|n}_{\mathrm{susu}}) \rightsquigarrow$ a "not-a-hook" relation.

The machine spits this out

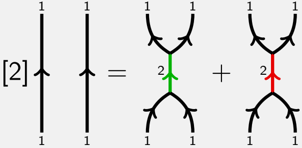
The (braided) monoidal, $\mathbb{C}_q$-linear category $N|M$-$\mathbf{Web}_{\mathrm{gr}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}$ and $\vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$. The *$N|M$-web space* $\mathrm{Hom}_{N|M\text{-}\mathbf{Web}_{\mathrm{gr}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by $N|M$-webs between $\vec{k}, \vec{l}$ modulo isotopies and:

$\mathfrak{gl}_m + \mathfrak{gl}_n$ "ladder" relations : same as before, but now in red as well!

Dumbbell relation :

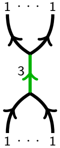


"Not-a-hook" relation : $e_q(\mathrm{box}_{N+1,M+1}) = 0.$

Exempli gratia

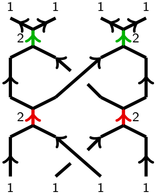
The "not-a-hook" relation kills the *Gyoja-Aiston idempotent* $e_q(\mathrm{box}_{N+1,M+1})$ for a box-shaped Young diagram with $N+1$ rows and $M+1$ columns. Examples:

$$e_q(\mathrm{box}_{3,1}) = e_q\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\right) = \frac{1}{[3]!}$$



$$= 0,$$

$$e_q(\mathrm{box}_{2,2}) = e_q\left(\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}\right) = \frac{1}{[2]^4}$$



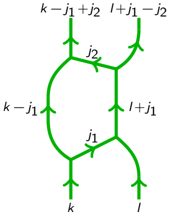
$$= 0.$$

The first is the exterior relation for green Temperley-Lieb-webs, the second is the relation found by Grant/Sartori for green $\mathfrak{gl}_{1|1}$-webs.

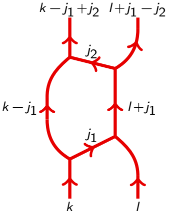
An almost perfect symmetry

Up to the exterior relations: $N|M\text{-}\mathbf{Web}_{\mathrm{gr}}$ is completely symmetric in green-red.
Only the *braiding* is slightly asymmetric, because $q \leftrightarrow q^{-1}$:

$$= (-1)^{k+kl} q^{k} \sum_{\substack{j_1, j_2 \geq 0 \\ j_1 - j_2 = k-l}} (-q)^{-j_1}$$



$$= (-1)^{k} q^{-k} \sum_{\substack{j_1, j_2 \geq 0 \\ j_1 - j_2 = k-l}} (-q)^{+j_1}$$



The ∞-webs space

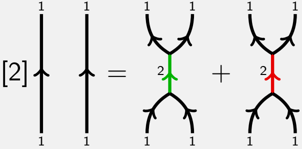
Define as before ∞-$\mathbf{Web}_{\mathrm{gr}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}$ and $\vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$. The ∞-*webs space* $\mathrm{Hom}_{\infty\text{-}\mathbf{Web}_{\mathrm{gr}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by ∞-webs between $\vec{k}, \vec{l}$ modulo isotopies and:

$\mathfrak{gl}_m + \mathfrak{gl}_n$ "ladder" relations : same as usual.

Dumbbell relation :



NO exterior relation :

$\neq 0$, if $k > N$.

The "big category"

For all $N \in \mathbb{N}$: there is a commuting diagram

$$\begin{array}{ccc} \infty\text{-}\mathbf{Web}_{\mathrm{gr}} & \xrightarrow{\pi^N_\infty} & N\text{-}\mathbf{Web}_{\mathrm{gr}} \\ \downarrow{\scriptstyle \Gamma^\infty} & & \downarrow{\scriptstyle \Gamma} \\ \check{\mathbf{H}} & \xrightarrow[\pi^N]{} & \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}} \end{array}$$

Here $\check{\mathbf{H}}$ is an *idempotented version of the Hecke algebroid* $\mathbf{H}$ and π^N is the full functor induced by *q-Schur-Weyl duality*:

$$\Phi^N_{q\mathrm{SW}} \colon H_K(q) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_N)}((\mathbb{C}^N_q)^{\otimes K}), \text{ if } N \geq K.$$

Theorem

$\Gamma^\infty \colon \infty\text{-}\mathbf{Web}^{\oplus}_{\mathrm{gr}} \to \check{\mathbf{H}}$ is an equivalence of (braided) monoidal categories.

An application: the HOMFLY-PT symmetry

Let $\mathcal{K}$ be a framed, oriented, colored knot $\mathcal{K}$. Associate to it the *colored HOMFLY-PT polynomial* $\mathcal{P}^{a,q}(\mathcal{K}(\lambda)) \in \mathbb{C}_q(a)$. The colors λ are Young diagrams.

The colored HOMFLY-PT polynomial can be defined from **H** and thus, from ∞-$\mathbf{Web}_{\mathrm{gr}}$. Since ∞-$\mathbf{Web}_{\mathrm{gr}}$ is symmetric in green-red and the braiding is symmetric in green-red under $q \leftrightarrow q^{-1}$:

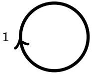
Corollary(of the green↔red symmetry)

The colored HOMFLY-PT polynomial satisfies

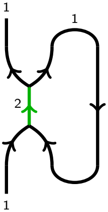
$$\mathcal{P}^{a,q}(\mathcal{K}(\lambda)) = (-1)^{co}\mathcal{P}^{a,q^{-1}}(\mathcal{K}(\lambda^{\mathrm{T}})),$$

where *co* is some constant. Similar for links.

To evaluate closed diagrams one only needs three extra rules:

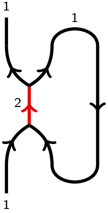


$$= \frac{a - a^{-1}}{q - q^{-1}},$$



$$= \frac{aq^{-1} - a^{-1}q}{q - q^{-1}}$$

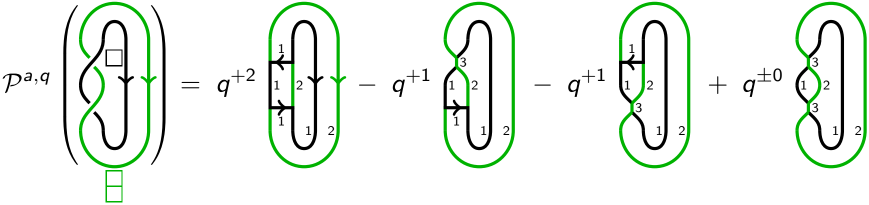
,

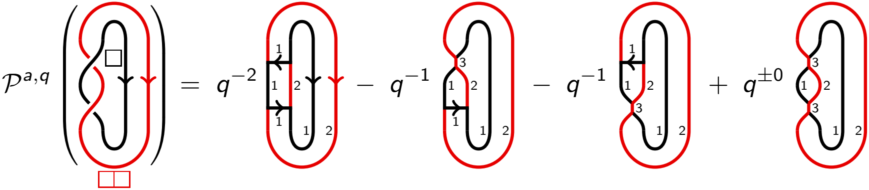


$$= \frac{aq - a^{-1}q^{-1}}{q - q^{-1}}$$

Green↔red and $q \leftrightarrow q^{-1}$ gives the “same” result (up to a sign).

Exempli gratia: the Hopf link $\mathcal{H}(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square) \leftrightarrow \mathcal{H}(\square\square, \square)$





I do not have tenure. So I have to bore you a bit more.

Some additional remarks.

- Homework: feed the machine with your favorite duality (e.g. Howe dualities in other types) and see what it spits out.
- The whole approach seems to be amenable to categorification.
- Relations to categorifications of the Hecke algebra using Soergel bimodules or category $\mathcal{O}$ need to be worked out.
- This could lead to a categorification of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{m|n})$ (since the "complicated" super relations are build in the calculus).
- A "green-red-foamy" approach could shed additional light on colored Khovanov-Rozansky homologies.
- The symmetry of the HOMFLY-PT polynomial holds (probably) for the homologies as well: maybe this can be proven by categorifying our approach.

There is still much to do...

Thanks for your attention!