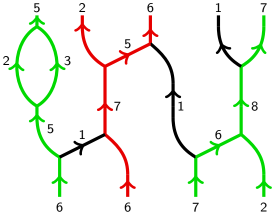
From dualities to diagrams

Or: the diagrammatic presentation machine

Daniel Tubbenhauer



Joint work with David Rose, Pedro Vaz and Paul Wedrich

May 2015

1. Exterior $\mathfrak{gl}_N$-web categories
   - Graphical calculus via Temperley-Lieb diagrams
   - Its cousins: the N-webs
   - Proof? Skew quantum Howe duality!
2. Symmetric $\mathfrak{gl}_2$-web categories
   - More cousins: the green 2-webs
   - Proof? Symmetric quantum Howe duality!
3. Exterior-symmetric $\mathfrak{gl}_N$-web categories
   - Even more cousins: the green-red N-webs
   - Proof? Super quantum Howe duality!
   - Super-Super duality and even more cousins

The 2-web space

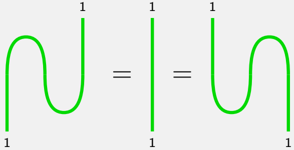
Definition(Rumer-Teller-Weyl 1932)

The 2-*web space* $\mathrm{Hom}_{2\text{-}\mathbf{Web}_{\mathrm{g}}}(b,t)$ is the free $\mathbb{C}_q = \mathbb{C}(q)$-vector space generated by non-intersecting arc diagrams with b, t bottom/top boundary points modulo:

- The *circle removal*:

$$= -q - q^{-1} = -[2]$$

- The *isotopy relations*:

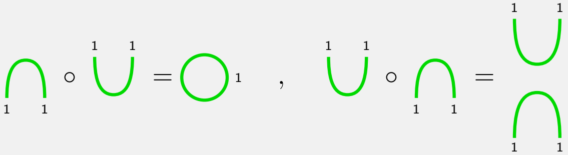


The 2-web category

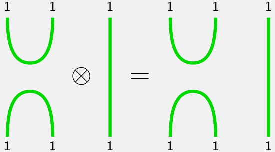
Definition(Kuperberg 1995)

The 2-*web category* 2-$\mathbf{Web}_{\mathrm{g}}$ is the (braided) monoidal, $\mathbb{C}_q$-linear category with:

- Objects are vectors $\vec{k} = (1, \ldots, 1)$ and morphisms are $\mathrm{Hom}_{2\text{-}\mathbf{Web}_{\mathrm{g}}}(\vec{k}, \vec{l})$.
- Composition $\circ$:



- Tensoring $\otimes$:



If you do not like quantum groups: $q = 1$ is fine for today

Recall that $\mathfrak{gl}_2$ is generated by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad H_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad ,$$

The elements of $\mathbf{U}(\mathfrak{gl}_2)$ are polynomials in $E, F, H_1, H_2, H = H_1 - H_2$ modulo

$$EF - FE = H,\ HE = EH + 2E,\ HF = FH + 2F.$$

The elements of $\mathbf{U}_q(\mathfrak{gl}_2)$ are polynomials in $E, F, L_{1,2}^{\pm 1}, K = L_1 L_2^{-1}$ modulo

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}},\ KE = q^2 EK,\ KF = q^{-2} FK.$$

Roughly: $K = q^H$ and $\lim_{q \to 1} \mathbf{U}_q(\mathfrak{gl}_2) = \mathbf{U}(\mathfrak{gl}_2)$.

Diagrams for intertwiners

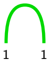
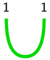
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_2)$-intertwiners

$$\mathrm{cap}\colon \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \to \mathbb{C}_q \quad \text{and} \quad \mathrm{cup}\colon \mathbb{C}_q \to \mathbb{C}_q^2 \otimes \mathbb{C}_q^2,$$

projecting $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$ onto $\mathbb{C}_q$ respectively embedding $\mathbb{C}_q$ into $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$.

Let $\mathfrak{gl}_2$-$\mathbf{Mod}_e$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\mathbb{C}_q^2$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{g}} \to \mathfrak{gl}_2\text{-}\mathbf{Mod}_e$:

- On objects: $\vec{k} = (1, \dots, 1)$ is send to $(\mathbb{C}_q^2)^{\otimes k} = \mathbb{C}_q^2 \otimes \cdots \otimes \mathbb{C}_q^2$.
- On morphisms:

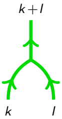

$\mapsto \mathrm{cap}$,

$\mapsto \mathrm{cup}$

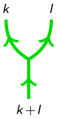
Theorem(Folklore)

$\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{g}}^{\oplus} \to \mathfrak{gl}_2\text{-}\mathbf{Mod}_e$ is an equivalence of (braided) monoidal categories.

The main step beyond $\mathfrak{gl}_2$: trivalent vertices

An *N-web* is an oriented, labeled, trivalent graph locally made of

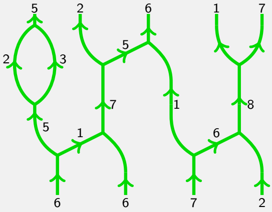
$\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{s}_{k+l}^{k,l} =$ 

$k, l, k+l \in \mathbb{N}$

(and no pivotal things today).

Example



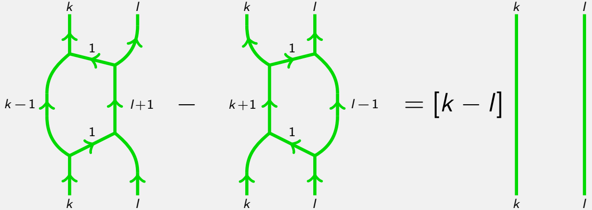
Let us try the same for $\mathfrak{gl}_N$: the N-web space

Define the (braided) monoidal, $\mathbb{C}_q$-linear category N-$\mathbf{Web}_{\mathrm{g}}$ by using:

Definition(Cautis-Kamnitzer-Morrison 2012)

The *N-web space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathrm{g}}}(\vec{k},\vec{l})$ is the free $\mathbb{C}_q$-vector space generated by N-webs with $\vec{k}$ and $\vec{l}$ at the bottom and top modulo isotopies and:

- "$\mathfrak{gl}_m$ *ladder*" relations like



- The *exterior relations*:

$= 0 \quad , \quad \text{if } k > N.$

Diagrams for intertwiners - Part 2

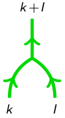
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_N)$-intertwiners

$$\mathrm{m}_{k,l}^{k+l}\colon \Lambda_q^k\mathbb{C}_q^N \otimes \Lambda_q^l\mathbb{C}_q^N \to \Lambda_q^{k+l}\mathbb{C}_q^N \quad \text{and} \quad \mathrm{s}_{k+l}^{k,l}\colon \Lambda_q^{k+l}\mathbb{C}_q^N \to \Lambda_q^k\mathbb{C}_q^N \otimes \Lambda_q^l\mathbb{C}_q^N$$

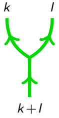
given by projection and inclusion again.

Let $\mathfrak{gl}_N$-$\mathbf{Mod}_e$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\Lambda_q^k\mathbb{C}_q^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e$:

- On objects: $\vec{k} = (k_1, \ldots, k_m)$ is send to $\Lambda_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \Lambda_q^{k_m}\mathbb{C}_q^N$.
- On morphisms:



$\mapsto \mathrm{m}_{k,l}^{k+l}$,



$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem(Cautis-Kamnitzer-Morrison 2012)

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}}^{\oplus} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e$ is an equivalence of (braided) monoidal categories.

“Howe” to prove this?

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_m)$ and $\mathbf{U}_q(\mathfrak{gl}_N)$ on

$$\begin{aligned}\textstyle\bigwedge_q^K(\mathbb{C}_q^m \otimes \mathbb{C}_q^N) &\cong \bigoplus_{k_1+\dots+k_m=K} (\textstyle\bigwedge_q^{k_1}\mathbb{C}_q^N \otimes \dots \otimes \bigwedge_q^{k_m}\mathbb{C}_q^N) \\ &\cong \bigoplus_{l_1+\dots+l_N=K} (\textstyle\bigwedge_q^{l_1}\mathbb{C}_q^m \otimes \dots \otimes \bigwedge_q^{l_N}\mathbb{C}_q^m)\end{aligned}$$

introduce an $\mathbf{U}_q(\mathfrak{gl}_m)$-action f on the first term with $\vec{k}$-weight space $\bigwedge_q^{\vec{k}}\mathbb{C}_q^N$.

In particular, there is a functorial action

$$\Phi^m_{\mathrm{skew}} \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e,$$
$$\vec{k} \mapsto \textstyle\bigwedge_q^{\vec{k}}\mathbb{C}_q^N, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{gl}_N\text{-}\mathbf{Mod}_e}(\bigwedge_q^{\vec{k}}\mathbb{C}_q^N, \bigwedge_q^{\vec{l}}\mathbb{C}_q^N).$$

Howe: Φ^m_{skew} is full. Or in words: all relations in $\mathfrak{gl}_N$-$\mathbf{Mod}_e$ follow from the ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ and the ones in the kernel of Φ^m_{skew}.

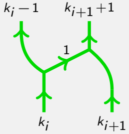
Define the diagrams to make this work

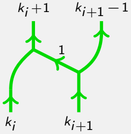
Theorem(Cautis-Kamnitzer-Morrison 2012)

Define $N\text{-}\mathbf{Web}_{\mathrm{g}}$ such there is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_q(\mathfrak{gl}_m) & \xrightarrow{\Phi^m_{\mathrm{skew}}} & \mathfrak{gl}_N\text{-}\mathbf{Mod}_e \\ & {\scriptstyle \Upsilon^m}\searrow \quad \nearrow{\scriptstyle \Gamma} & \\ & N\text{-}\mathbf{Web}_{\mathrm{g}} & \end{array}$$

with

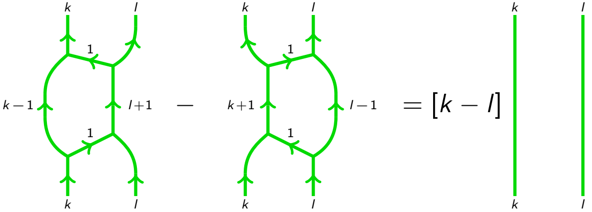
$\Upsilon^m(F_i 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^m(E_i 1_{\vec{k}}) \mapsto$ 

Υ^m induces the "$\mathfrak{gl}_m$ ladder" relations, $\ker(\Upsilon^m)$ gives the exterior relations.

Exempli gratia

The "$\mathfrak{gl}_m$ ladder" relation



is just

$$EF1_{\vec{k}} - FE1_{\vec{k}} = [k-l]1_{\vec{k}}.$$

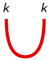
The exterior relations are a diagrammatic version of

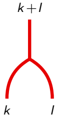
$$\Lambda_q^{>N}\mathbb{C}_q^N \cong 0.$$

The symmetric story is easier in some sense...

An 2-*web* is a labeled, trivalent graph locally made of

$\mathrm{cap}_k =$ 

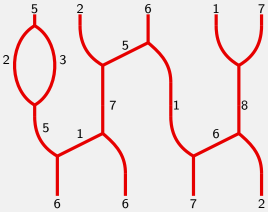
, $\mathrm{cup}_k =$ 

, $\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{s}_{k+l}^{k,l} =$ 

Up to sign issues that I ignore today!

Example

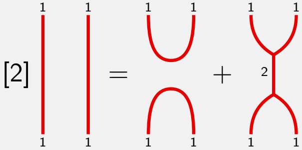


Define the (braided) monoidal, $\mathbb{C}_q$-linear category 2-$\mathbf{Web}_{\mathrm{r}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}^n_{\geq 0}$ and $\vec{l} \in \mathbb{Z}^{n'}_{\geq 0}$. The 2-*web space* $\mathrm{Hom}_{2\text{-}\mathbf{Web}_{\mathrm{r}}}(\vec{k},\vec{l})$ is the free $\mathbb{C}_q$-vector space generated by 2-webs between $\vec{k}$ and $\vec{l}$ modulo isotopies and:

- The "$\mathfrak{gl}_n$ *ladder*" relations again!
- A circle evaluation and the *dumbbell relation*:



- But *no(!)* relation of the form

$= 0$, if $k > N$.

Diagrams for intertwiners - Part 3

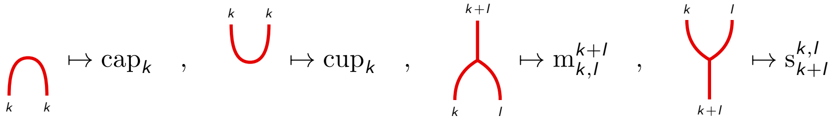
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_2)$-intertwiners

$$\mathrm{cap}_k\colon \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^k\mathbb{C}_q^2 \to \mathbb{C}_q \quad , \quad \mathrm{m}_{k,l}^{k+l}\colon \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2 \to \mathrm{Sym}_q^{k+l}\mathbb{C}_q^2$$
$$\mathrm{cup}_k\colon \mathbb{C}_q \to \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^k\mathbb{C}_q^2 \quad , \quad \mathrm{s}_{k+l}^{k,l}\colon \mathrm{Sym}_q^{k+l}\mathbb{C}_q^2 \to \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2$$

(guess where they come from...)

Let $\mathfrak{gl}_2$-$\mathbf{Mod}_s$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\mathrm{Sym}_q^k\mathbb{C}_q^N$. Define a functor $\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{r}} \to \mathfrak{gl}_2\text{-}\mathbf{Mod}_s$:

- On objects: $\vec{k} = (k_1, \ldots, k_n)$ is send to $\mathrm{Sym}_q^{k_1}\mathbb{C}_q^2 \otimes \cdots \otimes \mathrm{Sym}_q^{k_n}\mathbb{C}_q^2$.
- On morphisms:



Theorem

$\Gamma\colon 2\text{-}\mathbf{Web}_{\mathrm{r}}^{\oplus} \to \mathfrak{gl}_2\text{-}\mathbf{Mod}_s$ is an equivalence of (braided) monoidal categories.

“Howe” to prove this?

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{gl}_N)$ on

$$\mathrm{Sym}_q^K(\mathbb{C}_q^n \otimes \mathbb{C}_q^N) \cong \bigoplus_{k_1+\cdots+k_n=K} (\mathrm{Sym}_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_n}\mathbb{C}_q^N)$$
$$\cong \bigoplus_{l_1+\cdots+l_N=K} (\mathrm{Sym}_q^{l_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{l_N}\mathbb{C}_q^n)$$

introduce an $\mathbf{U}_q(\mathfrak{gl}_n)$-action f on the first term with $\vec{k}$-weight space $\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^N$.

In particular, there is a functorial action

$$\Phi^n_{\mathrm{sym}} \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_n) \to \mathfrak{gl}_2\text{-}\mathbf{Mod}_s,$$
$$\vec{k} \mapsto \mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_n)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{gl}_2\text{-}\mathbf{Mod}_s}(\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \mathrm{Sym}_q^{\vec{l}}\mathbb{C}_q^2).$$

Howe: Φ^n_{sym} is full. Or in words: all relations in $\mathfrak{gl}_2$-$\mathbf{Mod}_s$ follow from the ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$ and the ones in the kernel of Φ^n_{sym}.

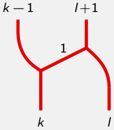
Copy-paste

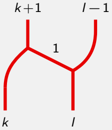
Theorem

Define 2-$\mathbf{Web}_{\mathrm{r}}$ such that there is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_q(\mathfrak{gl}_n) & \xrightarrow{\Phi^n_{\mathrm{sym}}} & \mathfrak{gl}_2\text{-}\mathbf{Mod}_s \\ & {\scriptstyle \Upsilon^n}\searrow \quad \nearrow{\scriptstyle \Gamma} & \\ & 2\text{-}\mathbf{Web}_{\mathrm{r}} & \end{array}$$

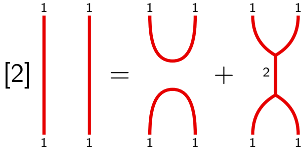
with

$\Upsilon^n(F_i 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^n(E_i 1_{\vec{k}}) \mapsto$ 

Υ^n induces the "$\mathfrak{gl}_n$ ladder" relations, $\ker(\Upsilon^n)$ gives the circle/dumbbell relation.

The dumbbell relation



is a diagrammatic version of

$$\mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \cong \mathbb{C}_q \oplus \mathrm{Sym}_q^2 \mathbb{C}_q^2.$$

No relations of the form

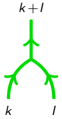
$$= 0 \quad , \quad \text{if } k > N,$$

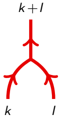
because

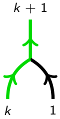
$$\mathrm{Sym}_q^{>N} \mathbb{C}_q^N \not\cong 0.$$

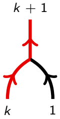
Could there be a pattern?

An green-red N-web is a colored, labeled, trivalent graph locally made of

$\mathrm{m}_{k,l}^{k+l} =$ 

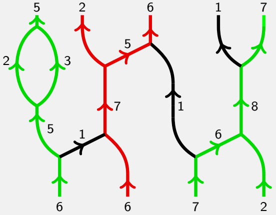
, $\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{m}_{k,1}^{k+l} =$ 

, $\mathrm{m}_{k,1}^{k+l} =$ 

And of course splits and some mirrors as well!

Example



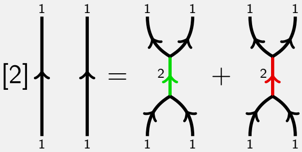
The green-red N-web category

Define the (braided) monoidal, $\mathbb{C}_q$-linear category $N\text{-}\mathbf{Web}_{\mathrm{gr}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}, \vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$. The green-red *N-web space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathrm{gr}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by N-webs between $\vec{k}$ and $\vec{l}$ modulo isotopies and:

- The "$\mathfrak{gl}_m + \mathfrak{gl}_n$ ladder" relations.
- The dumbbell relation:



- The *exterior relations*:

$= 0$, if $k > N$.

Diagrams for intertwiners - Part 4

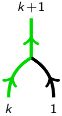
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_N)$-intertwiners

$$\mathrm{m}_{k,1}^{k+1}\colon \textstyle\bigwedge_q^k \mathbb{C}_q^N \otimes \mathbb{C}_q^N \to \bigwedge_q^{k+1}\mathbb{C}_q^N \quad \text{and} \quad \mathrm{m}_{k,1}^{k+1}\colon \mathrm{Sym}_q^k \mathbb{C}_q^N \otimes \mathbb{C}_q^N \to \mathrm{Sym}_q^{k+1}\mathbb{C}_q^N$$

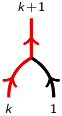
plus others as before.

Let $\mathfrak{gl}_N$-$\mathbf{Mod}_{\mathrm{es}}$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\bigwedge_q^k\mathbb{C}_q^N, \mathrm{Sym}_q^k\mathbb{C}_q^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$:

- On objects: $\vec{k} = (k_1, \dots, k_{m+n})$ is send to $\bigwedge_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_{m+n}}\mathbb{C}_q^N$.
- On morphisms:



$\mapsto \mathrm{m}_{k,1}^{k+1}$,



$\mapsto \mathrm{m}_{k,1}^{k+1}$, $\cdots$

Theorem

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}}^{\oplus} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$ is an equivalence of (braided) monoidal categories.

Super $\mathfrak{gl}(m|n)$

Definition

The *quantum general linear superalgebra* $\mathbf{U}_q(\mathfrak{gl}(m|n))$ is generated by $L_i^{\pm 1}$ and F_i, E_i subject the some relations, most notably, the *super relations*:

$$F_m^2 = 0 = E_m^2 \quad , \quad \frac{L_m L_{m+1}^{-1} - L_m^{-1} L_{m+1}}{q - q^{-1}} = F_m E_m + E_m F_m,$$

$$\begin{aligned}[2] F_m F_{m+1} F_{m-1} F_m = & F_m F_{m+1} F_m F_{m-1} + F_{m-1} F_m F_{m+1} F_m \\ & + F_{m+1} F_m F_{m-1} F_m + F_m F_{m-1} F_m F_{m+1} \text{ (plus an E version).}\end{aligned}$$

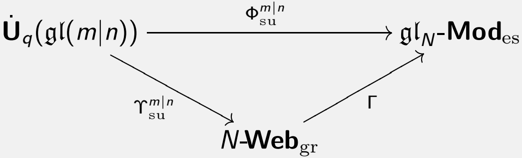
There is a Howe pair $(\mathbf{U}_q(\mathfrak{gl}(m|n)), \mathbf{U}_q(\mathfrak{gl}_N))$ with $\vec{k} = (k_1, \dots, k_{m+n})$-weight space under the $\mathbf{U}_q(\mathfrak{gl}(m|n))$-action on $\bigwedge_q^K (\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^N)$ given by

$$\textstyle\bigwedge_q^{k_1} \mathbb{C}_q^N \otimes \cdots \bigwedge_q^{k_m} \mathbb{C}_q^N \otimes \mathrm{Sym}_q^{k_{m+1}} \mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_{m+n}} \mathbb{C}_q^N.$$

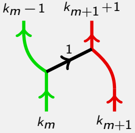
Define the diagrams to make this work

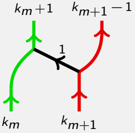
Theorem

Define N-$\mathbf{Web}_{\mathrm{gr}}$ such there is a commutative diagram



with

$\Upsilon_{\mathrm{su}}^{m|n}(F_m 1_{\vec{k}}) \mapsto$ 

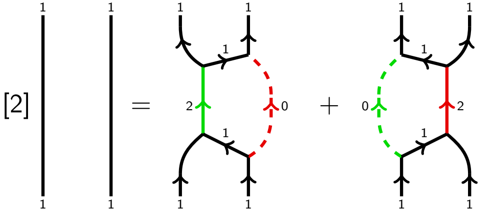
, $\Upsilon_{\mathrm{su}}^{m|n}(E_m 1_{\vec{k}}) \mapsto$ 

$\Upsilon_{\mathrm{su}}^{m|n}$ induces the "$\mathfrak{gl}(m|n)$ ladder" relations, $\ker(\Upsilon_{\mathrm{su}}^{m|n})$ gives the exterior relations.

Exempli gratia

The dumbbell relation is the super commutator relation:

$$[2]1_{(1,1)} = F_m E_m 1_{(1,1)} + E_m F_m 1_{(1,1)}$$



$$\mathbb{C}_q^N \otimes \mathbb{C}_q^N \cong \Lambda_q^2 \mathbb{C}_q^N \oplus \mathrm{Sym}_q^2 \mathbb{C}_q^N.$$

All other super relations are consequences!

Another meal for our machine

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}(m|n))$ and $\mathbf{U}_q(\mathfrak{gl}(N|M))$ on

$$\begin{aligned}\Lambda_q^K(\mathbb{C}_q^{m|n}\otimes\mathbb{C}_q^{N|M}) &\cong \bigoplus_{k_1+\cdots+k_n=K}(\Lambda_q^{\vec{k}_0}\mathbb{C}_q^{N|M}\otimes\mathrm{Sym}_q^{\vec{k}_1}\mathbb{C}_q^{N|M})\\ &\cong \bigoplus_{l_1+\cdots+l_N=K}(\Lambda_q^{\vec{l}_0}\mathbb{C}_q^{m|n}\otimes\mathrm{Sym}_q^{\vec{l}_1}\mathbb{C}_q^{m|n})\end{aligned}$$

introduce an $\mathbf{U}_q(\mathfrak{gl}(m|n))$-action f with $\vec{k}$-weight space $\Lambda_q^{\vec{k}_0}\mathbb{C}_q^{N|M}\otimes\mathrm{Sym}_q^{\vec{k}_1}\mathbb{C}_q^{N|M}$.

In particular, there is a functorial action

$$\begin{gathered}\Phi_{\mathrm{susu}}^{m|n}\colon \dot{\mathbf{U}}_q(\mathfrak{gl}(m|n))\to\mathfrak{gl}(N|M)\text{-}\mathbf{Mod}_{\mathrm{es}},\\ \vec{k}\mapsto\Lambda_q^{\vec{k}_0}\mathbb{C}_q^{N|M}\otimes\mathrm{Sym}_q^{\vec{k}_1}\mathbb{C}_q^{N|M},\quad \text{etc..}\end{gathered}$$

Howe: $\Phi_{\mathrm{susu}}^{m|n}$ is full. Or in words: all relations in $\mathfrak{gl}(N|M)$-$\mathbf{Mod}_{\mathrm{es}}$ follow from the ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}(m|n))$ and the ones in the kernel of $\Phi_{\mathrm{susu}}^{m|n}$.

The definition of the diagrams is already determined

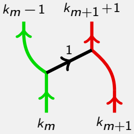
Theorem

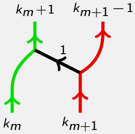
Define $N|M\text{-}\mathbf{Web}_{\mathrm{gr}}$ such there is a commutative diagram

$$\dot{\mathbf{U}}_q(\mathfrak{gl}(m|n)) \xrightarrow{\Phi^{m|n}_{\mathrm{susu}}} \mathfrak{gl}(N|M)\text{-}\mathbf{Mod}_{\mathrm{es}}$$

$$\Upsilon^{m|n}_{\mathrm{susu}} : \dot{\mathbf{U}}_q(\mathfrak{gl}(m|n)) \to N|M\text{-}\mathbf{Web}_{\mathrm{gr}}, \qquad \Gamma : N|M\text{-}\mathbf{Web}_{\mathrm{gr}} \to \mathfrak{gl}(N|M)\text{-}\mathbf{Mod}_{\mathrm{es}}$$

with

$\Upsilon^{m|n}_{\mathrm{susu}}(F_m 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^{m|n}_{\mathrm{susu}}(E_m 1_{\vec{k}}) \mapsto$ 

$\Upsilon^{m|n}_{\mathrm{susu}}$ induces "$\mathfrak{gl}(m|n)$ ladder" relations, $\ker(\Upsilon^{m|n}_{\mathrm{susu}})$ gives a "not-a-hook" relation.

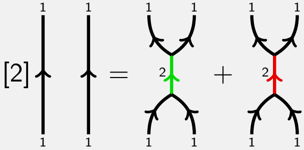
The machine spits this out

The (braided) monoidal, $\mathbb{C}_q$-linear category $N|M\text{-}\mathbf{Web}_{\mathrm{gr}}$ by using:

Definition

Given $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}$ and $\vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$. The *$N|M$-web space* $\mathrm{Hom}_{N|M\text{-}\mathbf{Web}_{\mathrm{gr}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by $N|M$-webs between $\vec{k}, \vec{l}$ modulo isotopies and:

- The "$\mathfrak{gl}_m + \mathfrak{gl}_n$ ladder" relations.
- The dumbbell relation:



- The *not-a-hook relations* (given by killing an idempotent corresponding to a box-shaped Young diagram).

Some concluding remarks

- Taking $N, M \to \infty$, one obtains a diagrammatic presentation ∞-$\mathbf{Web}_{\mathrm{gr}}$ of some form of the Hecke algebroid. Roughly: the machine spits it out, if you feed it with Schur-Weyl duality.
- ∞-$\mathbf{Web}_{\mathrm{gr}}$ is completely symmetric in green-red which allows us to prove a symmetry of HOMFLY-PT polynomials

$$\mathcal{P}^{a,q}(\mathcal{L}(\vec{\lambda})) \;=\; (-1)^{co}\, \mathcal{P}^{a,q^{-1}}(\mathcal{L}(\vec{\lambda}^{\,\mathrm{T}})).$$

  diagrammatically.
- Homework: feed the machine with you favorite duality (e.g. Howe dualities in other types) and see what it spits out.
- Everything is amenable to categorification!

There is still much to do...

Thanks for your attention!