Two-block Springer fibers and Springer representations in types C & D

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$GL_m(\mathbb{C})$-conjugacy classes of
nilpotent endomorphisms of $\mathbb{C}^m$

partitions λ of m

1:1

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$$\begin{tikzcd}[column sep=small] & \text{partitions } \lambda \text{ of } m \arrow[dl, leftrightarrow, "1{:}1"'] \arrow[dr, leftrightarrow, "1{:}1"] & \\ \begin{array}{c} GL_m(\mathbb{C})\text{-conjugacy classes of} \\ \text{nilpotent endomorphisms of } \mathbb{C}^m \end{array} & & \begin{array}{c} \text{complex, finite-dimensional,} \\ \text{irreducible } S_m\text{-representations} \\ \text{(up to isomorphism)} \end{array} \end{tikzcd}$$

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deeper/direct connection?

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$$\mathcal{O}_\lambda \rightsquigarrow \mathcal{B}^\lambda_{GL_m} \text{ Springer fiber}$$

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Definition (Springer fiber of type A)

$x\colon \mathbb{C}^m \to \mathbb{C}^m$ nilpotent endomorphism of Jordan type λ

$$\mathcal{B}^\lambda_{GL_m} = \{\{0\} \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_m = \mathbb{C}^m \mid xF_i \subseteq F_{i-1}\}$$

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Theorem (Springer, 1978)

There exists a graded S_m-action on $H^*(\mathcal{B}^\lambda_{GL_m}, \mathbb{C})$ such that $H^{\mathrm{top}}(\mathcal{B}^\lambda_{GL_m}, \mathbb{C})$ is the irreducible S_m-representation labeled by λ. This yields a correspondence

$$\mathrm{Irr}^{\mathrm{f.d.}}_{\mathbb{C}}(S_m) \xrightarrow{1:1} \{\text{nilpotent endomorphisms of } \mathbb{C}^m\} \Big/ GL_m(\mathbb{C}).$$

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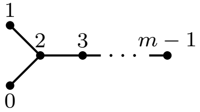
generators: $s_0, s_1, \dots, s_{m-1}$

relations: $(s_i s_j)^{m_{ij}} = e$

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i - j \text{ in} \\ 2 & \text{else.} \end{cases}$$



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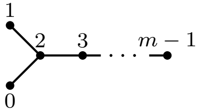

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subgroup generated by $s_1, \ldots, s_{m-1}$

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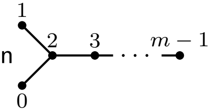

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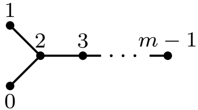

Theorem (Lusztig, 2004)

There exists an isomorphism of $\mathbb{C}[\mathcal{W}_{D_m}]$-modules

$$\underbrace{H^*(\mathcal{B}^{m,m}_{\mathrm{SO}_{2m}}, \mathbb{C})}_{\text{Springer representation}} \cong \underbrace{\mathbb{C} \otimes_{\mathbb{C}[S_m]} \mathbb{C}[\mathcal{W}_{D_m}]}_{\text{induced trivial module}}.$$

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- **Infinite-dimensional representation theory of Lie algebras.**

$$\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{so}_{2m}(\mathbb{C})) \qquad [M(\lambda)\colon L(\mu)] = \alpha_{\lambda,\mu}$$

(Kazhdan–Lusztig, Beilinson–Bernstein, Brylinski–Kashiwara, Elias–Williamson)

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Can we explicitly compute the $\alpha_{\lambda,\mu}$?

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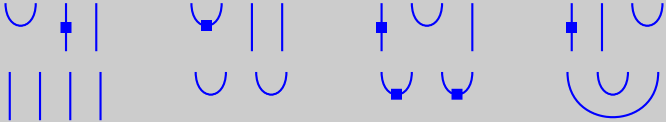
\{\text{Kazhdan-Lusztig basis } \underline{b}_\mu\} \arrow[r, "\psi", "\cong"'] & \left\{ \begin{array}{l} \text{cup diagrams on } m \text{ vertices,} \\ \#\left(\,|\!\!\blacksquare\,\right) + \#\left(\bigcup\right) \text{ even} \end{array} \right\}
\end{tikzcd}
$$

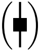
Example: (m=4)

$\vee\vee\vee\vee\,,\ \vee\vee\wedge\wedge\,,\ \vee\wedge\vee\wedge\,,\ \vee\wedge\wedge\vee\,,\ \wedge\vee\vee\wedge\,,\ \wedge\vee\wedge\vee\,,\ \wedge\wedge\vee\vee\,,\ \wedge\wedge\wedge\wedge$



$$\{\text{standard basis } b_\lambda\} \xrightarrow[\cong]{\phi} \left\{ \begin{array}{l} \{\wedge,\vee\}\text{-sequences,} \\ \text{length } m,\ \#(\wedge) \text{ even} \end{array} \right\}$$

$$\underline{b}_\mu = \textstyle\sum_\lambda \alpha_{\lambda,\mu} b_\lambda$$

$\{\text{Kazhdan-Lusztig basis } \underline{b}_\mu\} \xrightarrow[\cong]{\psi}$ { cup diagrams on m vertices, $\#($  $) + \#($  $)$ even }

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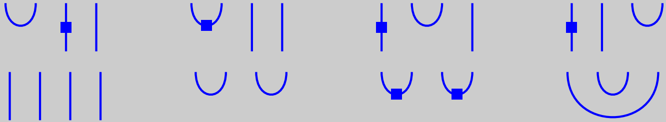

Theorem (Lejczyk–Stroppel, 2013)

$$\alpha_{\lambda,\mu} = \begin{cases} 1\,, & \text{if } \begin{array}{c}\phi(b_\lambda) \\ \psi(\underline{b}_\mu)\end{array} \text{ oriented,} \\ 0\,, & \text{else.} \end{cases}$$

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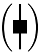
$\{\text{Kazhdan-Lusztig basis } \underline{b}_\mu\} \xrightarrow[\cong]{\psi} \Big\{$ cup diagrams on m vertices, $\#($  $) + \#($  $)$ even $\Big\} = C_{\mathrm{KL}}(m)$

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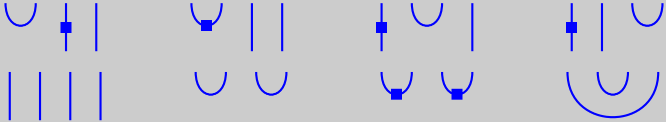

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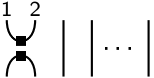
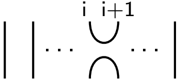
$e_0 = s_0 - 1 =$ 

$e_i = s_i - 1 =$ 

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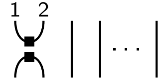
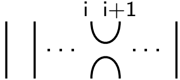
$e_0 = s_0 - 1 =$ 

$e_i = s_i - 1 =$ 

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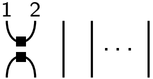
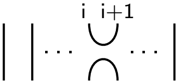
$e_0 = s_0 - 1 =$ 

$e_i = s_i - 1 =$ 

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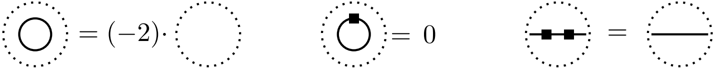

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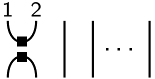
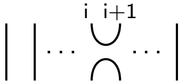
$e_0 = s_0 - 1 =$ 

$e_i = s_i - 1 =$ 

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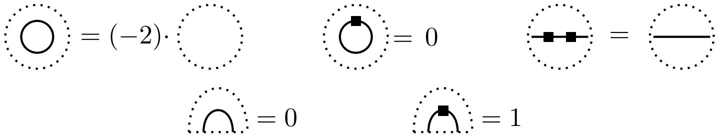
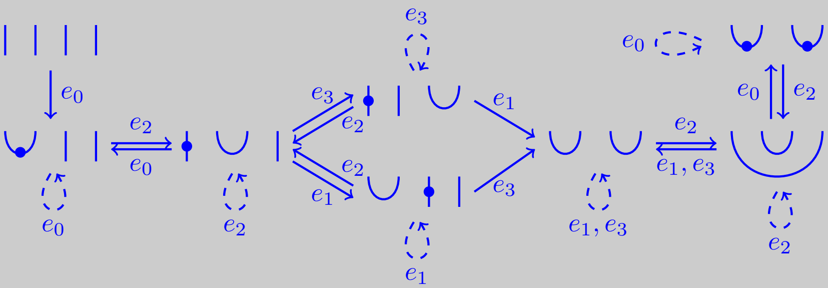


Example: $(m = 4)$



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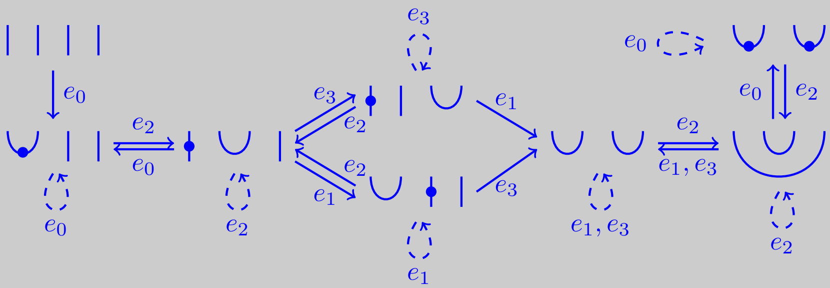

Remark

There exists a filtration

$$\{0\} \subseteq \mathbb{C}[C_{\mathrm{KL}}(m)]_{\lfloor \frac{m}{2} \rfloor} \subseteq \ldots \subseteq \mathbb{C}[C_{\mathrm{KL}}(m)]_n \subseteq \ldots \subseteq \mathbb{C}[C_{\mathrm{KL}}(m)]_0 = \mathbb{C}[C_{\mathrm{KL}}(m)]$$

of $\mathbb{C}[\mathcal{W}_{D_m}]$-modules, where

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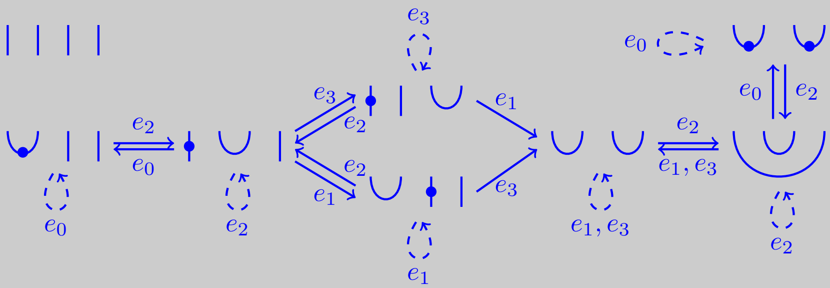

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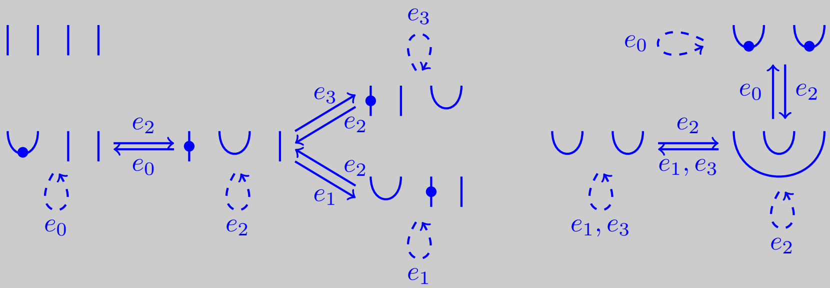

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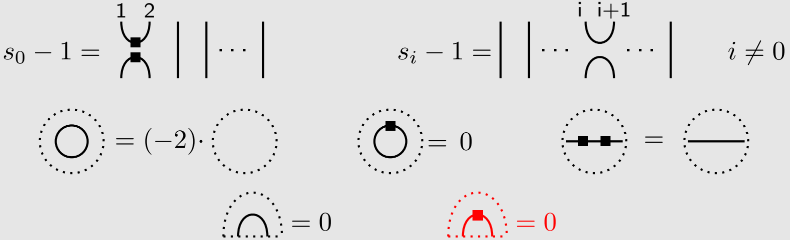
Remark (cont.)

The subquotients

$$\mathbb{C}[C_{\mathrm{KL}}(m)]_n \Big/ \mathbb{C}[C_{\mathrm{KL}}(m)]_{n+1} = \operatorname{span}_{\mathbb{C}}\{[\mathbf{a}] \mid \mathbf{a} \in C_{\mathrm{KL}}(m)\,,\ \#(\text{cups}) = n\}$$

are irreducible $\mathbb{C}[\mathcal{W}_{D_m}]$-modules with $\mathbb{C}[\mathcal{W}_{D_m}]$-action given by:



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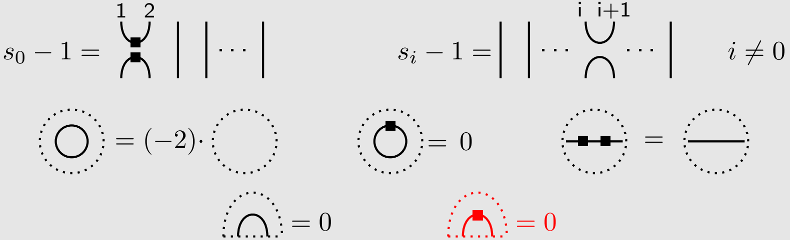

Moreover, we have an isomorphism of $\mathbb{C}[\mathcal{W}_{D_m}]$-modules

$$H^*(\mathcal{B}^{m,m}_{\mathrm{SO}_{2m}}, \mathbb{C}) \cong \bigoplus_n \mathbb{C}[C_{\mathrm{KL}}(m)]_n \Big/ \mathbb{C}[C_{\mathrm{KL}}(m)]_{n+1}\,.$$

Question

Is the cup diagram combinatorics describing $H^*(\mathcal{B}^{m,m}_{SO_{2m}}, \mathbb{C})$ already visible on the space $\mathcal{B}^{m,m}_{SO_{2m}}$? Does this tell us anything about the topology of $\mathcal{B}^{m,m}_{SO_{2m}}$ (or even more generally $\mathcal{B}^{2m-k,k}_{SO_{2m}}$)?

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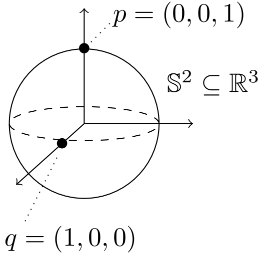

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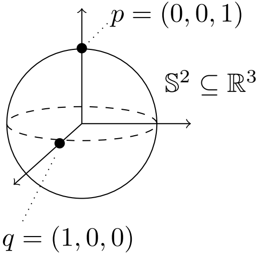

$C_{\mathrm{KL}}(m)$

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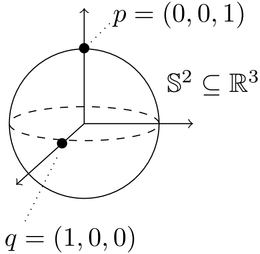

$$\underbrace{C_{2m-k,k}(m)}_{\text{diagrams with } \lfloor \frac{k}{2} \rfloor \text{ cups}} \subseteq C_{\mathrm{KL}}(m)$$

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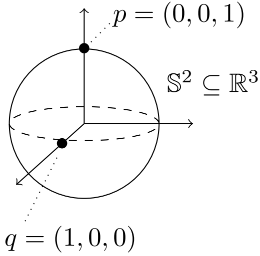

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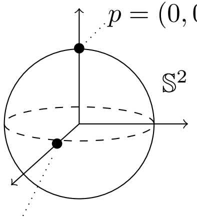

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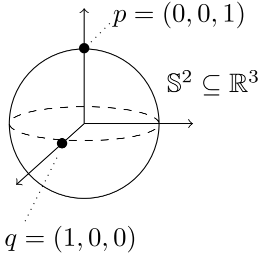

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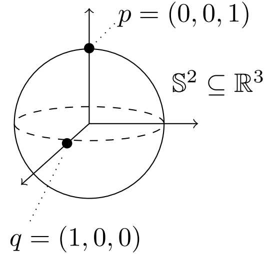

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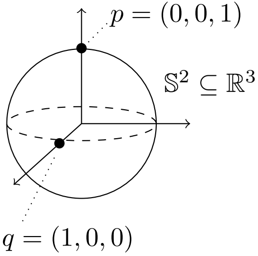

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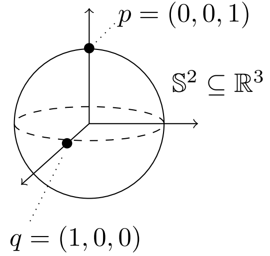

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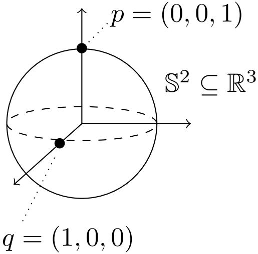
Is the cup diagram combinatorics describing $H^*(\mathcal{B}^{m,m}_{SO_{2m}}, \mathbb{C})$ already visible on the space $\mathcal{B}^{m,m}_{SO_{2m}}$? Does this tell us anything about the topology of $\mathcal{B}^{m,m}_{SO_{2m}}$ (or even more generally $\mathcal{B}^{2m-k,k}_{SO_{2m}}$)?



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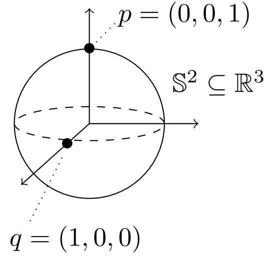

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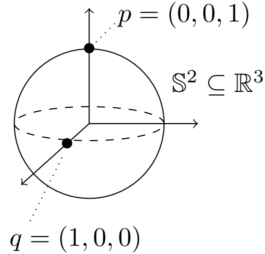

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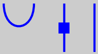
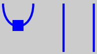
$\mathbf{a} =$  $\mathbf{b} =$  $\mathbf{c} =$  $\mathbf{d} =$ 

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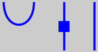
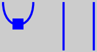
$\mathbf{a} =$ 
$\mathbf{b} =$ 
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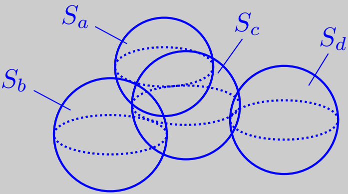
$\mathcal{S}^{5,3}_{SO_8} \cong$ 

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We have a "commutative diagram"

$$\begin{tikzcd}
& \mathcal{P}erv\left(\Upsilon^D_m\right) \arrow[rr, dashed, "\text{“(Stroppel-Webster)”}"] & & D^b\left(\mathsf{Coh}(Y^D_m)\right) & \\
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$$
\begin{tikzcd}
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action on $(\mathbb{S}^2)^m$:

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$$\begin{tikzcd}
H_*((\mathbb{S}^2)^m,\mathbb{C}) \arrow[r, phantom, "\circlearrowleft"] & \mathbb{C}[\mathcal{W}_{D_m}] \arrow[d, squiggly, "\text{action restricts}"] \\
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action restricts

(The component group for $\mathcal{B}^{m,m}_{SO_{2m}}$ is trivial.)

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Theorem (2016)

We have an isomorphism of $\mathbb{C}[\mathcal{W}_{D_m}]$-modules

$$H_{2n}(\mathcal{S}^{m,m}_{SO_{2m}}, \mathbb{C}) \xrightarrow{\cong} \mathbb{C}[C_{\mathrm{KL}}(m)]_n \Big/ \mathbb{C}[C_{\mathrm{KL}}(m)]_{n+1}\,,\ [C_{\mathbf{a}}] \mapsto [\mathbf{a}].$$

In particular,

$$H_*(\mathcal{S}^{m,m}_{SO_{2m}}, \mathbb{C}) \cong \mathbb{C}[C_{\mathrm{KL}}(m)] \cong \mathbb{C} \otimes_{\mathbb{C}[S_m]} \mathbb{C}[\mathcal{W}_{D_m}] \cong H^*(\mathcal{B}^{m,m}_{SO_{2m}}, \mathbb{C}).$$

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How can we reconstruct the component group action on the topological model? What is its diagrammatic description?

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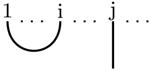
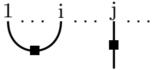
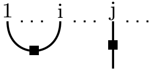
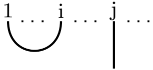
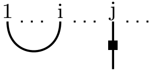
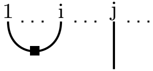
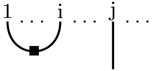
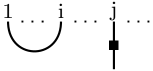
$\mathbf{a}$	$\mathbf{a}^*$
	
	
	
	

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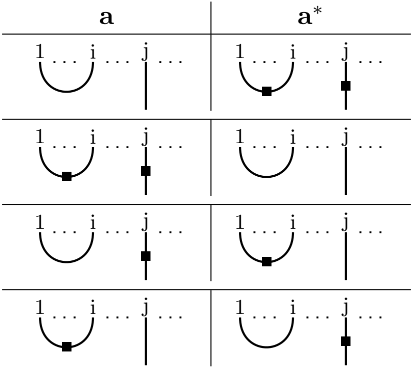

Theorem (2016)

The $\mathbb{Z}/2\mathbb{Z}$-action on $H_*(\mathcal{S}^{m,m}_{SO_{2m}}, \mathbb{C})$ (m odd) is given by

$$(-1) \cdot [C_{\mathbf{a}}] = \begin{cases} [C_{\mathbf{a}}] & \text{if } 1 \text{ is connected to a ray,} \\ [C_{\mathbf{a}^*}] & \text{if } 1 \text{ is conntected to a cup.} \end{cases}$$