EXERCISES 10: LECTURE ALGEBRAIC TOPOLOGY

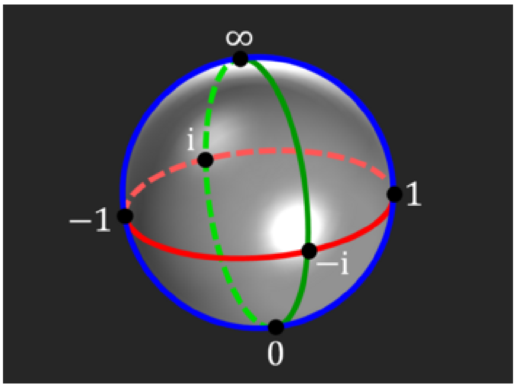
Exercise 1. The unitary group is $U(1) = S^1 \subset \mathbb{C}$ with multiplication being induced by $\mathbb{C}$. Note that $U(1)$ acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by scalar multiplication. Define the complex projective n-space as

$$\mathbb{C}P^n = S^{2n+1}/U(1).$$

(By convention, $\mathbb{C}P^0$ is a point.) Compute $H^*(\mathbb{C}P^n)$.

Addendum:

- https://en.wikipedia.org/wiki/Complex_projective_space#Introduction explains nicely why this is a "good definition" of complex projective spaces.
- Hint: The case $n = 1$ gives

$\mathbb{C}P^1 \cong$  $\cong S^2$

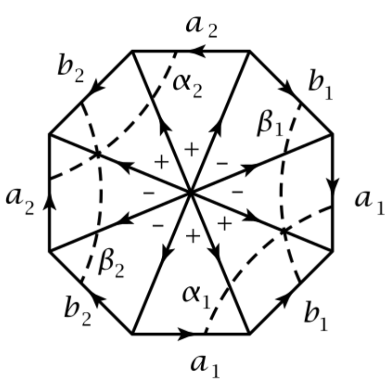
Exercise 2. Use the result from Exercise 1 to compute the cohomology ring $H^\bullet(\mathbb{C}P^n)$.

Hint: If you want to try Exercise 2 without solving Exercise 1, then you can use that $\mathbb{C}P^n$ is a cell complex with precisely one $2k$-cell for $k = 0, \dots, n$ and no other cells.

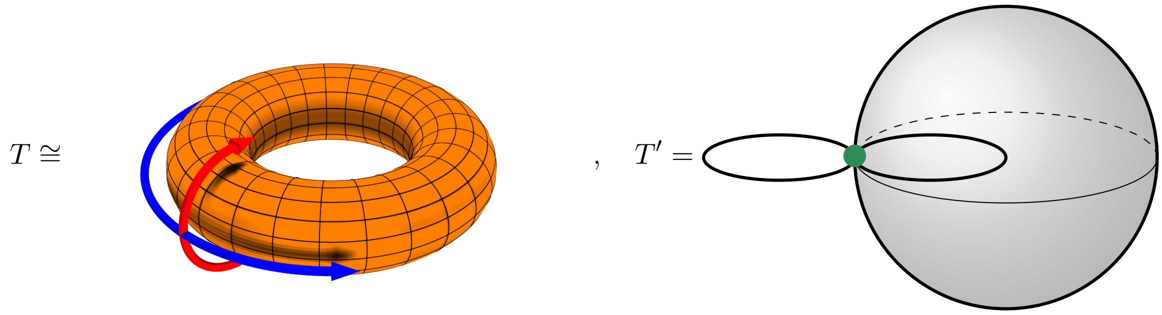
Exercise 3. Let $M_{g,0}$ be the surface of genus $g > 0$. Compute the cohomology ring $H^\bullet(M_{g,0})$.

Addendum:

- This is Example 3.7 in Hatcher.
- Hint: Use a "good" simplicial structure on these, *e.g.*

$M_{2,0} \rightsquigarrow$ 

Exercise 4. Given the torus $T \cong S^1 \times S^1$ and the space $T' = S^1 \vee S^1 \vee S^2$.



Compute the cohomology rings $H^\bullet$ for these two spaces, and compare with Exercise 2 from the previous exercise sheet.

- The exercises are optimal and not mandatory. Still, they are highly recommend.
- There will be 12 exercise sheets, all of which have four exercises.
- The sheets can be found on the homepage www.dtubbenhauer.com/lecture-algtop-2021.html.
- If not specified otherwise, spaces are topological space, maps are continuous *etc.*
- There might be typos on the exercise sheets, my bad, so be prepared.