EXERCISES 3: LECTURE ALGEBRAIC TOPOLOGY

Exercise 1. Show that for every group homomorphism

$$f\colon \pi_1(S^1) \to \pi_1(S^1)$$

there exists $g\colon S^1 \to S^1$ such that $f = g_*$.

Exercise 2. Show that there is a group isomorphism

$$\pi_1(X \times Y) \xrightarrow{\cong} \pi_1(X) \times \pi_1(Y), [f] \mapsto (p_*([f]), q_*([f]))$$

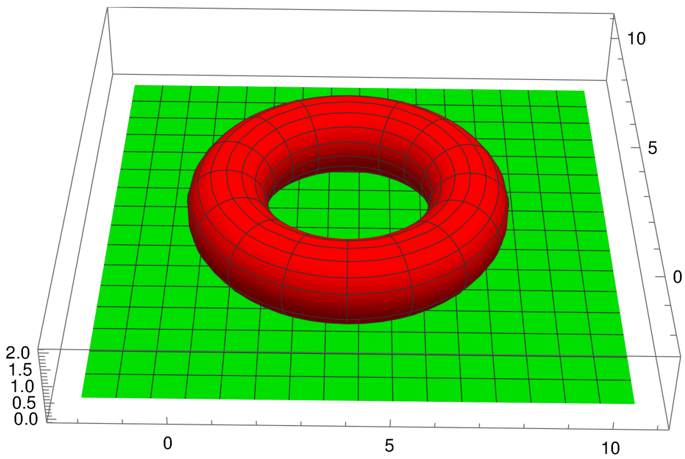
where p and q are the projections of $X \times Y$ onto its two factors, respectively.

(Note that $\pi_1(X)$ is a shorthand notation for $\pi_1(X, x_0)$, whenever X is path-connected. In particular, X and Y in this exercise are assumed to be path-connected.)

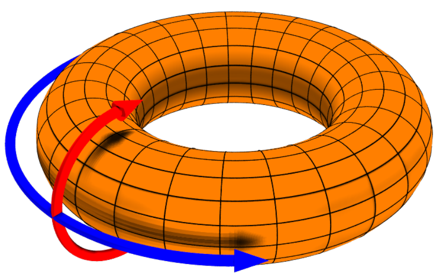
Exercise 3. Is the following true or false? For every map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ there exists (x, y) such that $f(x, y) = f(-x, -y)$.

Addendum:

- Reformulated: Does the Borsuk–Ulam theorem hold for the torus?
  https://en.wikipedia.org/wiki/Borsuk-Ulam_theorem
- Hint: Think of the torus T (red) as lying on the ground (green):



Exercise 4. Let T be the torus, and let l be the longitude and m be the meridian:

$T =$ 

1. Show that $\pi_1(T)$ is generated by (path corresponding to) l and m.
2. Show that $\pi_1(T)$ is commutative.
3. Show that $\pi_1(T) \xrightarrow[m \mapsto (0,1)]{l \mapsto (1,0)} \mathbb{Z}^2$ is a group isomorphism.

Addendum:

- Note that 3. $\Rightarrow$ 2. and 3. $\Rightarrow$ 1. (Can you see why?)
- Hint: www.youtube.com/watch?v=nLcr-DWVEto
- Exercise 2. looks related.

- The exercises are optimal and not mandatory. Still, they are highly recommend.
- There will be 12 exercise sheets, all of which have four exercises.
- The sheets can be found on the homepage www.dtubbenhauer.com/lecture-algtop-2021.html.
- If not specified otherwise, spaces are topological space, maps are continuous *etc.*
- There might be typos on the exercise sheets, my bad, so be prepared.