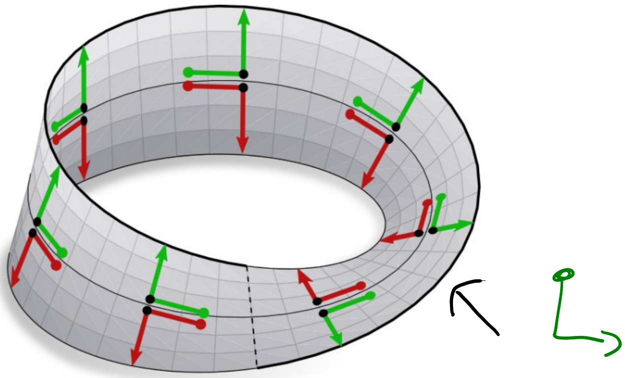
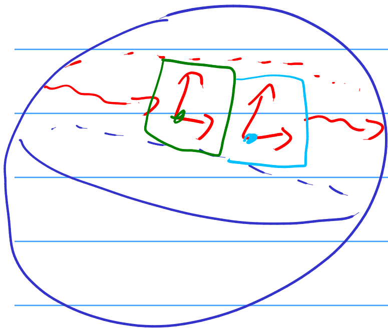
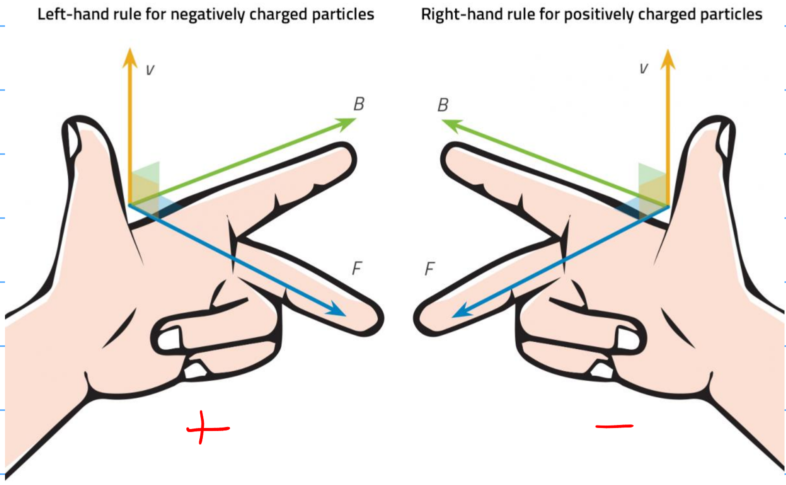




- Orientability of a manifold is a consistent choice of a coordinate system per point
- There are non-orientable manifolds
- What can homology say about orientability?

An orientation of $\mathbb{R}^n$ is a choice of a left- or right-handed coordinate system: A positive orientation is a basis that comes from bases change from the standard basis via a matrix of positive determinant; a negative orientation are those having a negative change-of-basis determinant.



$$\left.\begin{matrix} 1, 0, \dots \\ 0, 1, \dots \\ 0, 0, 1 \dots \end{matrix}\right\} B$$

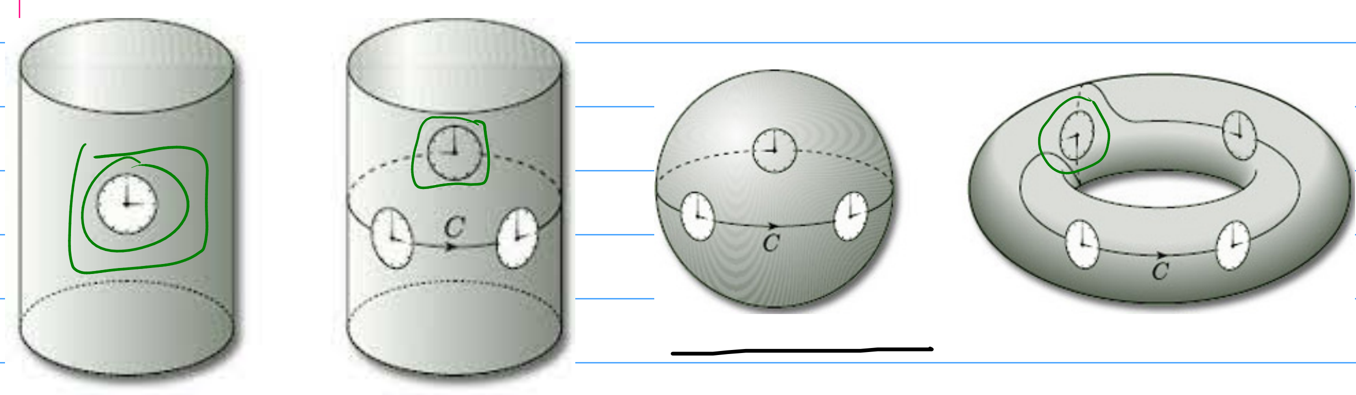
Base-change-matrix

$B \rightsquigarrow$ "favorite basis"

$A \in GL_n(\mathbb{R})$

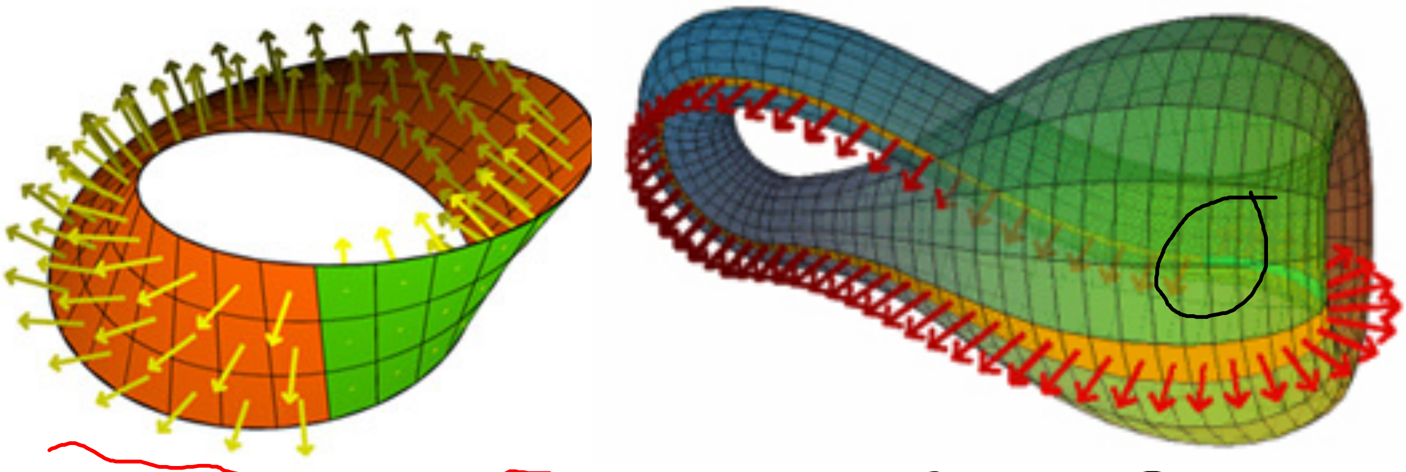
$\det(A) = \pm 1$

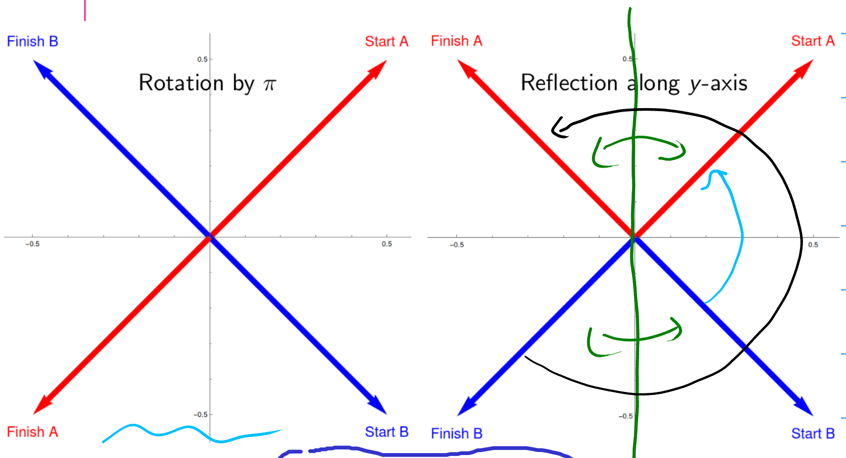
An orientation of a smooth manifold M is a continuous choice of an orientation of the tangent space T_xM.



(a) (b)

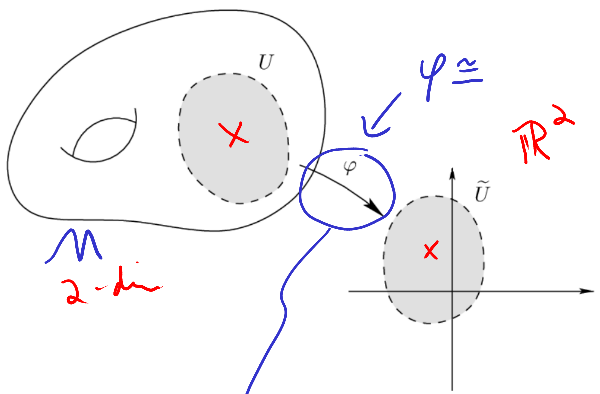
An orientation of a surface S is a choice of normal vector per point that continuously varies over S.





- An orientation should be preserved under rotation and translation and scaling
- An orientation should be reversed under reflection

"Def:" A structure on M is orientation at $x \in M$ if its reversed under reflection and preserved under rotation



$H_n(X,Y)$
homol. of $C(X/Y)$

- By local triviality of an n-manifold M one gets

$$H_n(M, M\setminus\{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

- Rotations/reflections give maps from $H_{n-1}(S^{n-1})$ to itself, satisfying

$$\text{Rotation}_*(\pm 1) = \pm 1 \qquad \text{Reflection}_*(\pm 1) = \mp 1$$

$H_n(M, M\setminus\{x\})$
$\simeq H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{\varphi(x)\})$
$\simeq H_{n-1}(S^{n-1}) \simeq \mathbb{Z}$

rot, refl. : $\mathbb{Z} \xrightarrow{\pm 1} \mathbb{Z}$

no smoothness condition

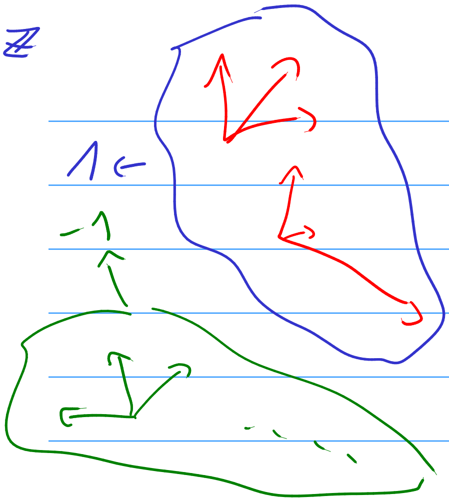
Let M be an n-manifold

- A local orientation at $x \in M$ is a choice $\alpha_x = \pm 1 \in H_n(M, M \setminus \{x\}) \simeq \mathbb{Z}$
- A (global) orientation is a consistent choice of α_x for all x, meaning:

$$\left\{ \begin{array}{c} \forall x \in M \; \exists \text{ open } U \cong D^n \subset \mathbb{R}^n \text{ containing } x \text{ such that} \\ \exists \alpha_U = \pm 1 \in H_n(M, M \setminus U) \cong \mathbb{Z} \text{ with} \\ \forall y \in U\colon (\iota_y)_*\colon H_n(M, M \setminus U) \to H_n(M, M \setminus \{y\}), \quad \alpha_U \mapsto \alpha_y \end{array} \right\}$$

where $\iota_y\colon (M, M \setminus U) \to (M, M \setminus \{y\})$ is the inclusion

- If an orientation exists for M, then M is called orientable



If M is orientable, then there are two different orientations?

If M is smooth, then homologically orient is the same as any of the "classical defs"

Main observation: $H_n(M, M-\{x\}) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \simeq \mathbb{Z}$

- The second point should be read as "Every x has a neighborhood in which the orientation is rotated or is translated or scaled but not reflected

  Compatibility condition formulated homologically

- The same definition works for homology with coefficients in an arbitrary PID

$$H_n(M, M-\{x\}, \overset{R}{\mathbb{Z}/2\mathbb{Z}}) \simeq \dots \simeq \overset{R}{\mathbb{Z}/2\mathbb{Z}}$$

$\pm 1 \leftrightsquigarrow$ choices of gens of R

$\rightsquigarrow$ R-orientable

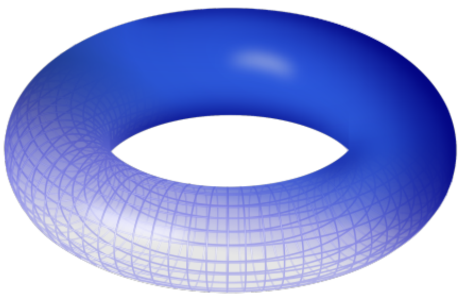
Example: ✓ Any M is $\mathbb{Z}/2\mathbb{Z}$-orientable

Theorem If M is a closed connected n-manifold, then

n-dim

$$H_n(M) \cong \begin{cases} \mathbb{Z} \text{ if } M \text{ is orientable} \\ 0 \text{ if } M \text{ is not orientable} \end{cases}$$

very simple condition! Cool!



$H_*(\text{torus}) \cong \mathbb{Z} \oplus t\mathbb{Z}^{\oplus 2} \oplus t^2\mathbb{Z}$

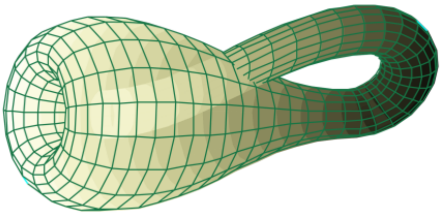
$\mathbb{R}P^2$

$\Rightarrow$ torus is orientable

$\mathbb{R}P^n$ is orientable $\Leftrightarrow$ n is odd

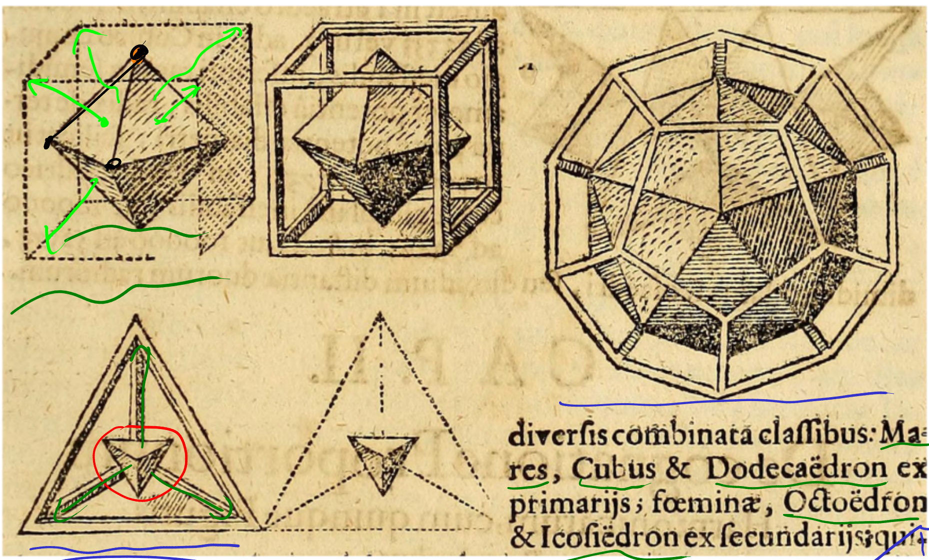
$\mathbb{R}P^n$: $\overset{3}{\mathbb{Z}} \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{2} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}}$

$H_2(\mathbb{R}P^2) = 0$



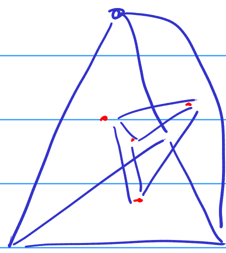
$H_*(\text{Klein bottle}) \cong \mathbb{Z} \oplus t(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus t^2 0$

Klein bottle is not orientable

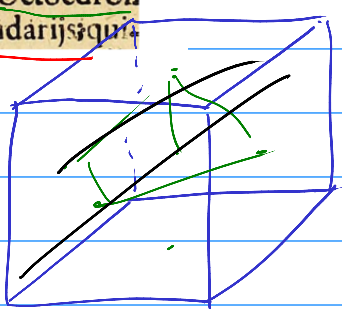


$H_3(\mathbb{R}P^3) \cong \mathbb{Z}$

↑ orientable!

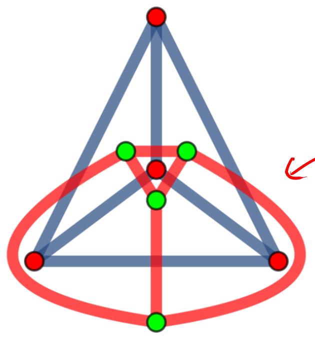


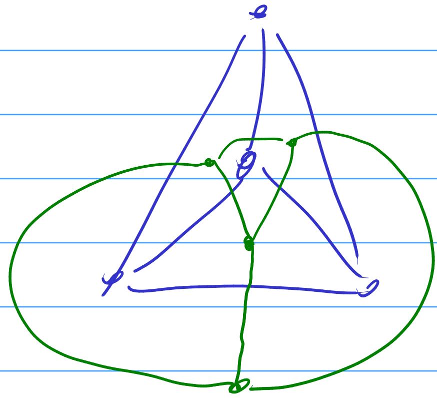
tetr. $\leftrightarrow$ tetra
cube $\leftrightarrow$ octa
Dode $\leftrightarrow$ Ico



- The dual of a tetrahedron is a tetrahedron
- The dual of a cube is a octahedron
- The dual of a dodecahedron is a icosahedron

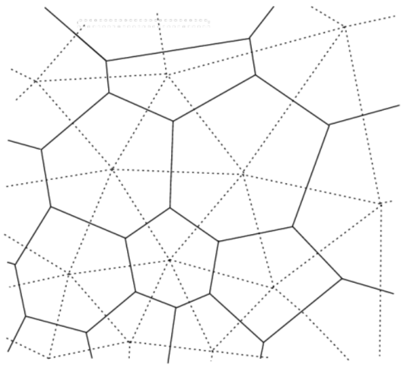
What does this mean?





The dual G^* of a plane graph G_* is obtained by reversing dimensions:

- G^* has a vertex for each face of G_*
- G^* has an edge for each edge of G_*; connecting adjacent faces
- G^* has a face for each vertex of G_*



- Similarly, for any cell complex X_* one can define a dual cell complex X^*

finite

- We have $\chi(X_*) = \pm\chi(X^*)$ since

$+0 \quad -1 \quad +2 \quad -3$ or $+0 \quad -1 \quad +2 \quad -3 \quad +4$

$-$ $\qquad$ $+$

If M is an orientable closed n-manifold, then for $0 \leq k \leq n$:

$$[M] \frown \_ : H^k(M) \xrightarrow{\cong} H_{n-k}(M)$$

$n \longleftrightarrow 0$

$n-1 \longleftrightarrow 1$

$\vdots \qquad \vdots$

- Here $\frown$ is the pairing

$$\_ \frown \_ : H_k(M) \times H^l(M) \to H_{k-l}(M), \sigma \frown \phi = \phi(\sigma|[v_0, \dots, v_l])\sigma|[v_l, \dots, v_k]$$

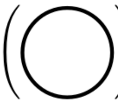
- There are many generalization, *e.g.* relaxing "orientable" or "closed"

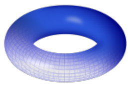
$[M] \in H_n(M)$ fundamental class

$\hookrightarrow \cong \mathbb{Z} \Leftrightarrow$ orientable

This implies that the Hilbert–Poincaré polynomial of M is palindromic:

rank

S^1 $P\left(\right.$  $\left.\right) = 1 + t \rightsquigarrow 1 \quad t$

T $P\left(\right.$  $\left.\right) = 1 + 2t + t^2 \rightsquigarrow 1 \quad 2t \quad t^2$

$$P\left(\mathbb{C}P^6\right) = 1 + t^2 + t^4 + t^6 \rightsquigarrow 1 \quad 0 \quad t^2 \quad 0 \quad t^4 \quad 0 \quad t^6$$

The universal coefficient theorem (UCT) for cohomology for all X and PID R:

$$0 \to \mathrm{Ext}\left(H_{k-1}(X), R\right) \to H^k(X, R) \to \hom\left(H_k(X), R\right) \to 0$$

is a split (non-naturally) short exact sequence

- Thus, in general

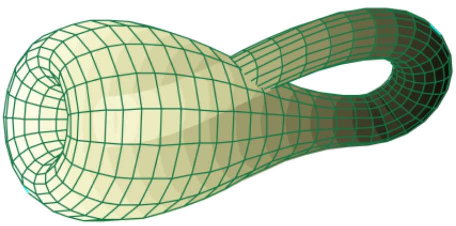
$$H^k(X) \cong \hom\left(H_k(X), \mathbb{Z}\right) \oplus \mathrm{Ext}\left(H_{k-1}(X), \mathbb{Z}\right)$$

- Ext vanishes over $\mathbb{Q}$ and $\hom\left(H_k(X), \mathbb{Q}\right) \cong H_k(X, \mathbb{Q})$ if finite, which implies

$$H_k(M, \mathbb{Q}) \cong H^k(M, \mathbb{Q})$$

- Paste this together with Poincaré duality:

$$H_k(M, \mathbb{Q}) \cong H^k(M, \mathbb{Q}) \cong H_{n-k}(M, \mathbb{Q})$$

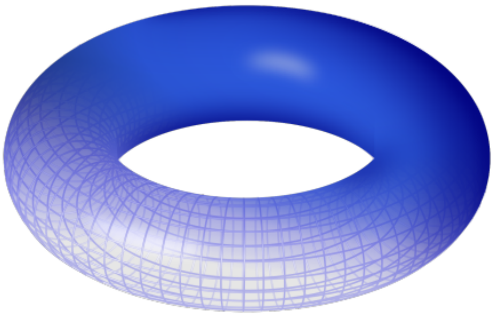


$$H_*(\text{Klein bottle}) \cong \mathbb{Z} \oplus t(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus t^2 0$$

t t

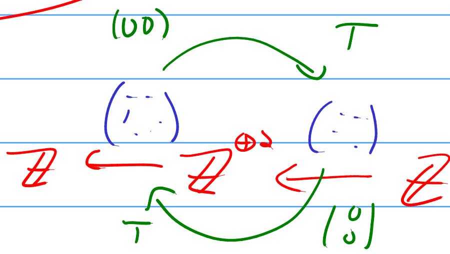
$$\mathbb{Z} \xleftarrow{(0\,2)} \mathbb{Z}^2 \xleftarrow{0} \mathbb{Z}$$

$$H^*(K) \cong \mathbb{Z} \oplus t\mathbb{Z} \oplus t^2\mathbb{Z}/2\mathbb{Z}$$



$$H_*(\text{torus}) \cong \mathbb{Z} \oplus t\mathbb{Z}^{\oplus 2} \oplus t^2\mathbb{Z}$$

$$H^*(\text{torus}) \cong \mathbb{Z} \oplus t\mathbb{Z}^{\oplus 2} \oplus t^2\mathbb{Z}$$



Let X be a closed oriented (smooth) manifold of dimension n. Let A and B be oriented smooth submanifolds of X of dimensions $n-i$ and $n-j$ respectively. Assume that A and B intersect transversely.

The images of A, B and $A \cap B$ under the inclusions into X define homology classes $[A], [B], [A \cap B]$. We denote their Poincaré duals by $[A]^*, [B]^*, [A \cap B]^*$. We now have:

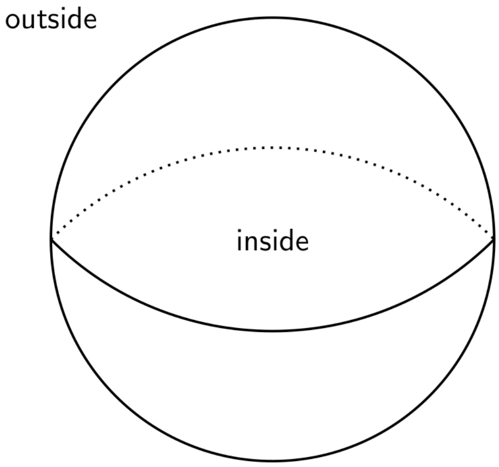
Theorem. Cup product is Poincaré dual to intersection:

$$[A]^* \smile [B]^* = [A \cap B]^*.$$

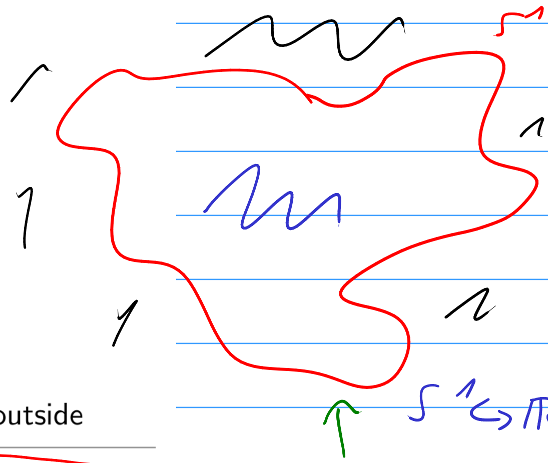
Catch. Not all X are closed oriented (smooth) manifold.

Catch. Not all generators of cohomology arise from submanifolds (although counterexamples are somewhat hard to come by).

Brouwer-Jordan



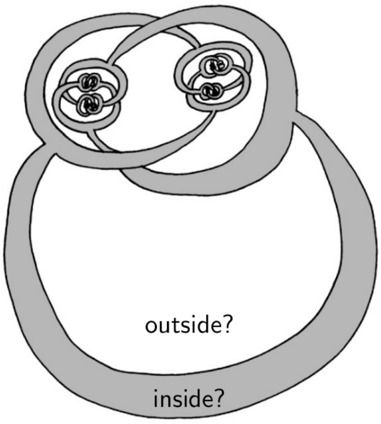
$S^1 \hookrightarrow \mathbb{R}^2$



$S^1 \hookrightarrow \mathbb{R}^2$

A sphere S^2 embedded in $\mathbb{R}^3$ divides $\mathbb{R}^3$ into an inside and an outside

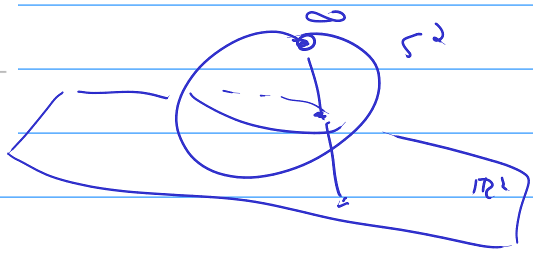
Formally $\mathbb{R}^3 \setminus \iota(S^2)$ is has two connected components for any $\iota\colon S^2 \hookrightarrow \mathbb{R}^3$



Alexander horned sphere

A sphere S^2 embedded in $\mathbb{R}^3$ divides $\mathbb{R}^3$ into an inside and an outside. Really?

The more one thinks about it, the less clear it becomes!



- We can replace $\mathbb{R}^3$ with S^3 Stereographic Projection
- The number of connected component of $S^3 \setminus \iota(S^2)$ is $\dim H_0(S^3 \setminus \iota(S^2))$
- Hence, reduced homology should satisfy

$$\dim \tilde{H}_0(S^3 \setminus \iota(S^2), \mathbb{Q}) = 1$$

- So we need to compute $\dim \tilde{H}_0(S^3 \setminus \iota(S^2), \mathbb{Q})$

If $\emptyset \subsetneq K \subsetneq S^n$ is a compact and locally contractible, then

$$\tilde{H}_i(S^n \setminus K) \xrightarrow{\cong} \tilde{H}^{n-i-1}(K)$$

Alexander duality

- This only depends on intrinsic properties of K $\iota(S^{n-1})$
- For $K = \iota(S^{n-1}) \cong S^{n-1}$ one gets

$$\tilde{H}_0(S^n \setminus S^{n-1}) \xrightarrow{\cong} \tilde{H}^{n-1}(S^{n-1}) \cong \mathbb{Z}$$

$$\tilde{H}_0(S^n \setminus S^{n-1}) \cong \tilde{H}^{n-0-1}(S^{n-1}) \cong \mathbb{Z}$$

- Thus, we get

$$\dim \tilde{H}_0(S^n \setminus \iota(S^{n-1}); \mathbb{Q}) = 1$$

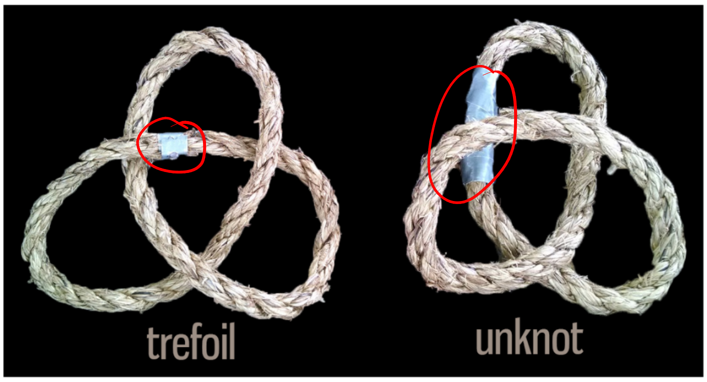
- This is a consequence of (the a bit more general)

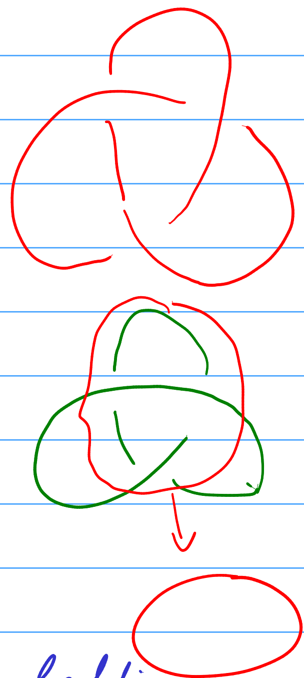
$$H_i(M, M \setminus K) \xrightarrow{\cong} H^{n-i}(K)$$

where M is closed orientable n-manifold and where $K \subset M$ is compact and locally contractible

Alexander duality

$\rightsquigarrow$ Proof Jordan-Brouwer theorem





- A knot K is an embedding $S^1 \hookrightarrow S^3 \rightsquigarrow$ thickened into a torus $\overline{K} \cong T$ A rope
- One gets

$$\tilde{H}_i(S^n \setminus \overline{K}) \xrightarrow{\cong} \tilde{H}^{n-i-1}(\overline{K}) \cong \tilde{H}^{n-i-1}(T)$$

- This does not depend on the embedding, so can not distinguish knots

$\rightsquigarrow$ homology can not distinguish embeddings