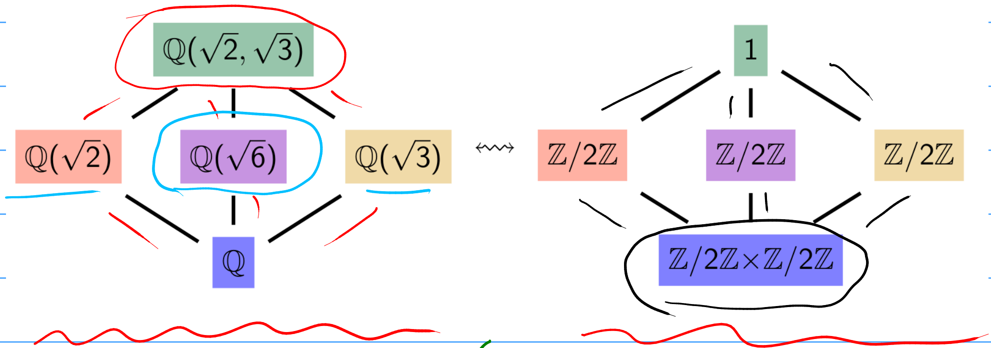
Covering space in topology $\Leftrightarrow$ field ext. in Galois theory

Field extensions and subgroups of the Galois group, *e.g.*



$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$

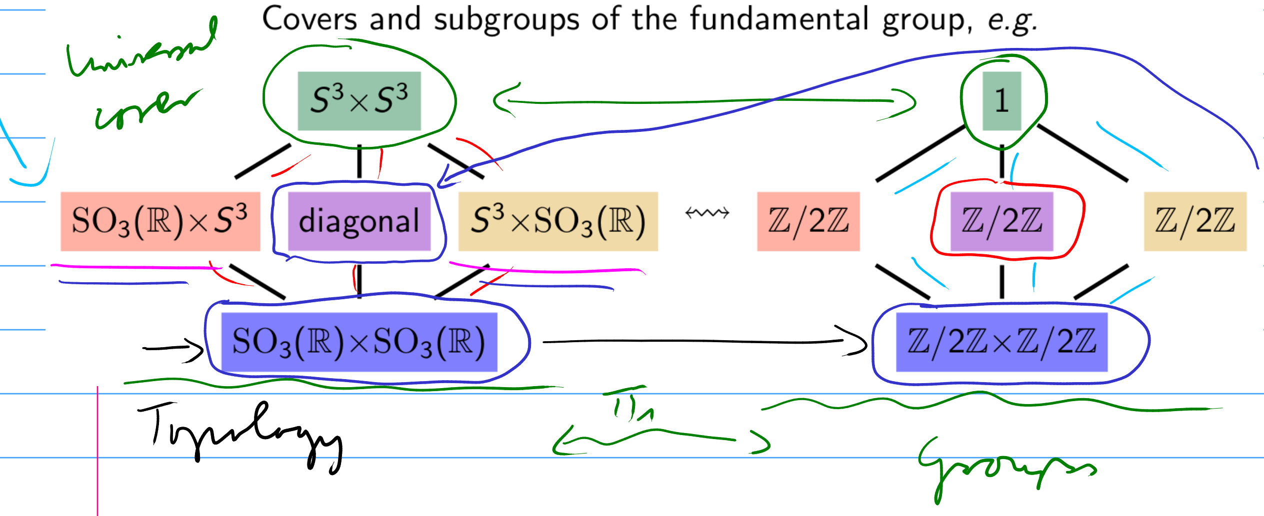
$(0,0)$

$(1,0)$

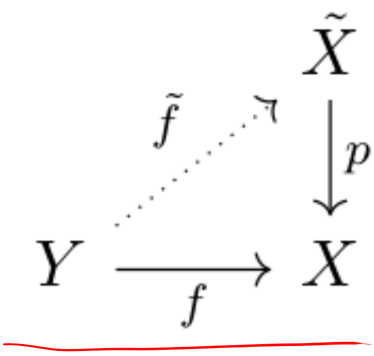
$(0,1)$

$(1,1)$

Covers and subgroups of the fundamental group, *e.g.*

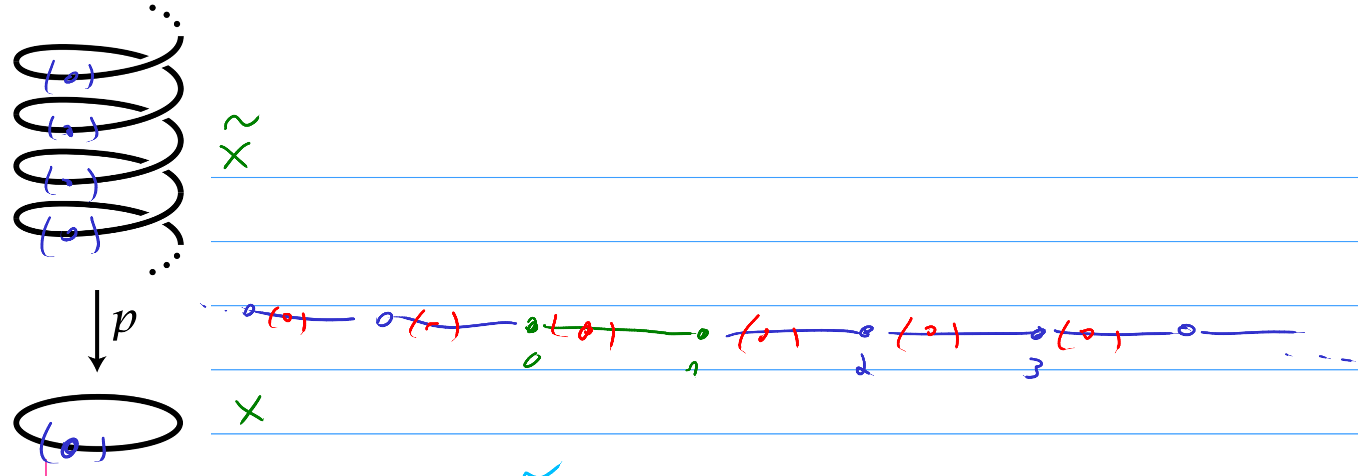


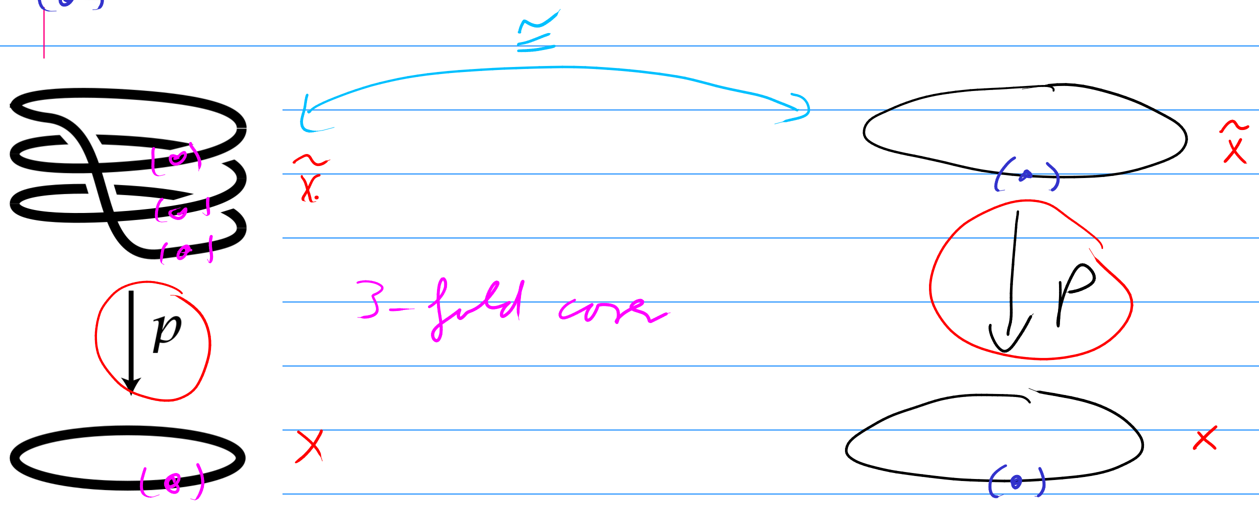
Covering $\tilde{X}$ of X is a pair $(\tilde{X}, p)$ $p: \tilde{X} \to X$ satisfying the open neighborhood condition



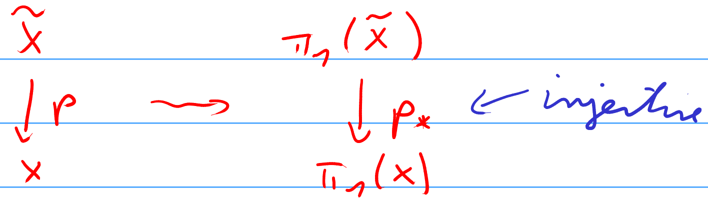
- every $x \in X$ has U open, $x \in U$ such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$ all of which are mapped homeo to U

Proposition 1.30. *Given a covering space $p:\tilde{X}\to X$, a homotopy $f_t:Y\to X$, and a map $\tilde{f}_0:Y\to\tilde{X}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t:Y\to\tilde{X}$ of $\tilde{f}_0$ that lifts f_t.*

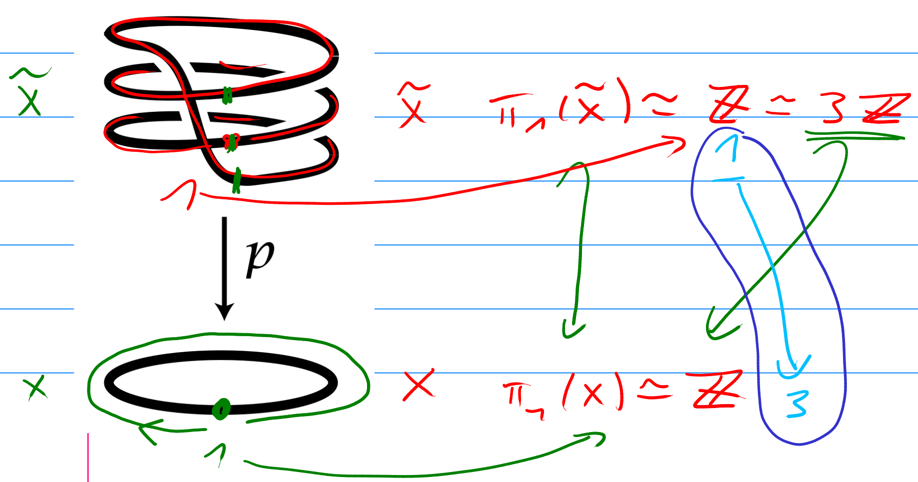


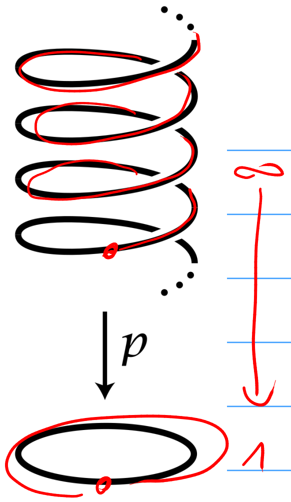


Proposition 1.31. *The map $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ induced by a covering space $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.*



⇝ $\pi_1(\tilde{X})$ is a subgroup of $\pi_1(X)$





$1 = \pi_1(\tilde{X})$

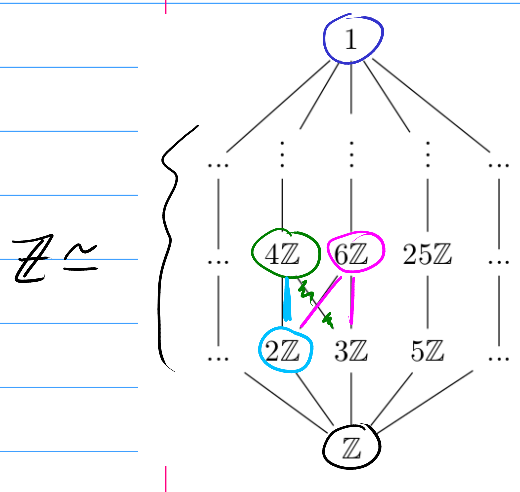
$\pi_1(X) \cong \mathbb{Z}$

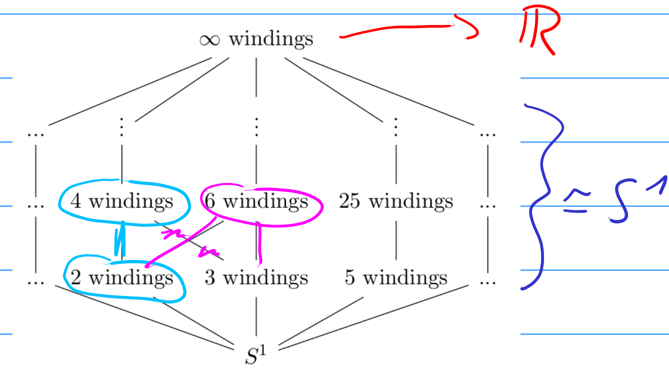
$\pi_1(X) / \pi_1(\tilde{X})$

Proposition 1.32. *The number of sheets of a covering space* $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ *with* X *and* $\tilde{X}$ *path-connected equals the index of* $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ *in* $\pi_1(X,x_0)$.

Every cover $\tilde{X} \to X \rightsquigarrow$ subgroup of $\pi_1(X)$

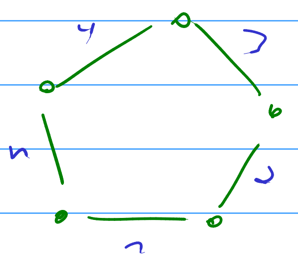
Question: What about the converse?



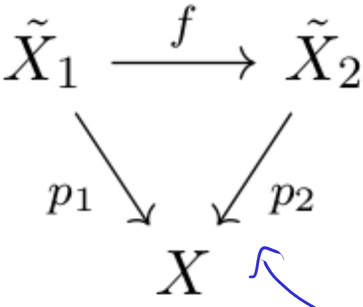


Algebra

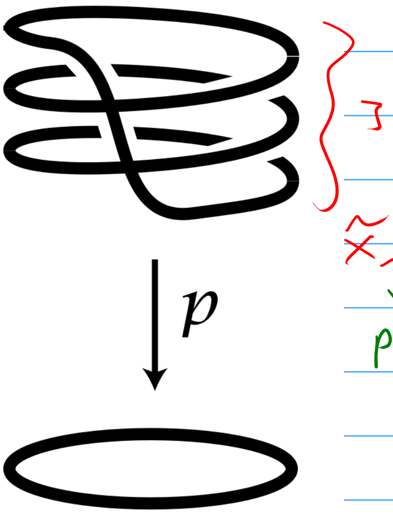
Topology

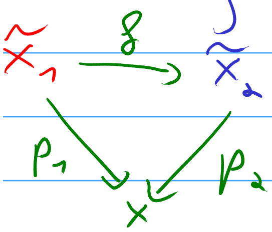


$n \longrightarrow \infty$



Def: $\tilde{X}_1 \xrightarrow{p_1} X$ cover, $\tilde{X}_2 \xrightarrow{p_2} X$ cover.
We say they are equivalent if $\exists f : \tilde{X}_1 \xrightarrow{\cong} \tilde{X}_2$ homeo. and that $p_2 \circ f = p_1$

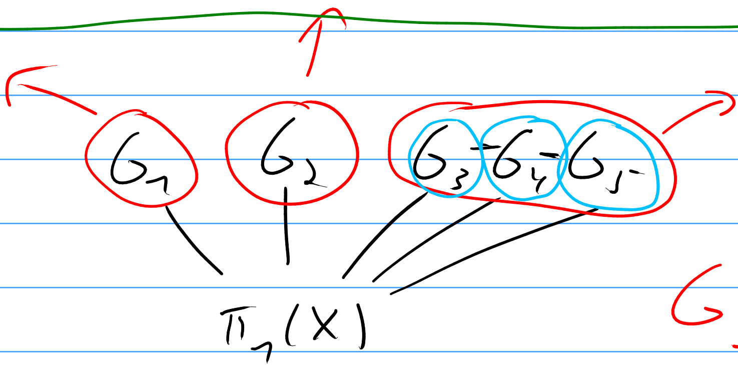




All f's will preserve the property of being a 3-fold cover under covering equivalence

Theorem 1.38. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ and the set of subgroups of $\pi_1(X,x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ to the covering space $(\tilde{X},\tilde{x}_0)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p:\tilde{X}\to X$ and conjugacy classes of subgroups of $\pi_1(X,x_0)$.*

Great! Galois for coverings



iso class of path-connected coverings

$G_3 = g G_4 g^{-1} = h G_5 h^{-1}$ for some $g, h \in \pi_1(X)$

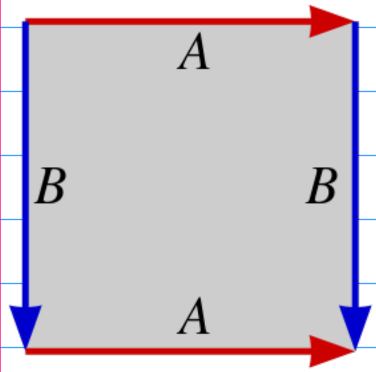
The one for the unique copy of the trivial group and it is the unique simply-connected, path connected cover of X

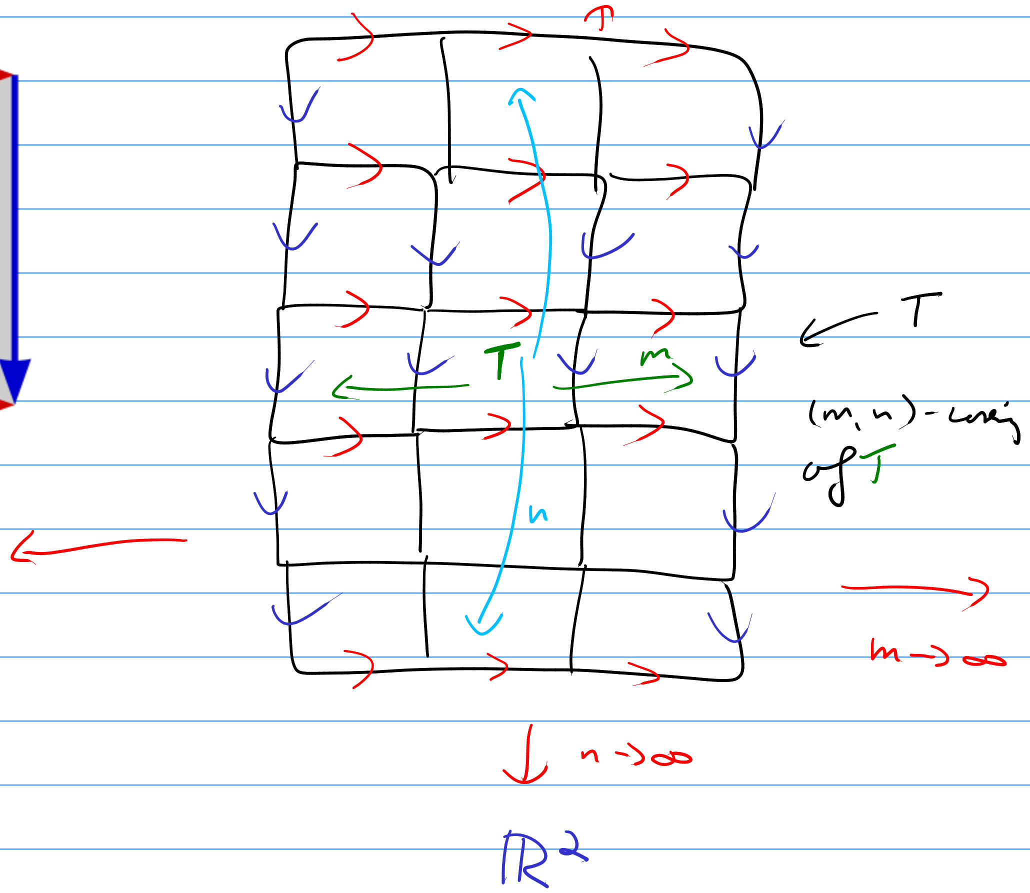
most of them are S^1

Example: $X = S^1 \rightsquigarrow \infty$ many covers, universal cover is $\mathbb{R}$

In the abelian case $g h g^{-1} = g g^{-1} h = h$

$T =$





Torus has two types of covers:

- T (m, n)-fold cover ∞-many
- $\mathbb{R}^2$ (universal cover)

Question: Are there any other covers?

Isomorphism classes of subgroups of $\pi_1(\text{torus}) \simeq \mathbb{Z}^2$ and associated $\tilde{X}$ up to $\cong$ of topological spaces:

(a) $\mathbb{Z}^2 \rightsquigarrow S^1 \times S^1$ ← $\mathbb{Z} \times \mathbb{Z}$

(b) $\mathbb{Z} \rightsquigarrow S^1 \times \mathbb{R}$ ← $\mathbb{Z} \times 1$

(c) $1 \rightsquigarrow \mathbb{R} \times \mathbb{R}$ ← 1×1

There are however ∞ many conjugacy classes of subgroups of $\pi_1(\text{torus}) \simeq \mathbb{Z}^2$

Compute: $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$

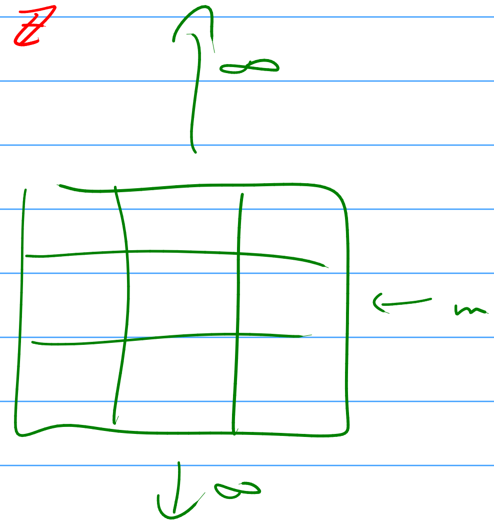
⇝ need to write down the subgroup lattice of $\mathbb{Z} \times \mathbb{Z}$

$$\mathbb{Z} \times \mathbb{Z}$$
$$\leq$$
$$m\mathbb{Z} \times n\mathbb{Z} \qquad (m, n)$$

$$1 \times \mathbb{Z} \cong 1 \times n\mathbb{Z}$$

$$m\mathbb{Z} \times 1 \cong \mathbb{Z} \times 1$$

$$1 \times 1 \cong 1$$



$m\mathbb{Z} \times n\mathbb{Z}$

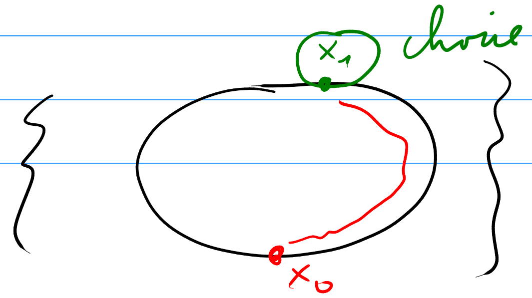
⇝ coverings of T

Theorem 1.38. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ and the set of subgroups of $\pi_1(X,x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ to the covering space $(\tilde{X},\tilde{x}_0)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p:\tilde{X}\to X$ and conjugacy classes of subgroups of $\pi_1(X,x_0)$.*

Given a path-connected, locally path-connected, semilocally simply-connected space X with a basepoint $x_0 \in X$, we are therefore led to define

$$\tilde{X} = \{\, [\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0 \,\}$$

How to go from a given subgroup to coverings?

$\tilde{X} =$ 

Given a set $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, let

$$U_{[\gamma]} = \{ [\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1) \}$$

⇝ Topology on $\tilde{X}$

$p : \tilde{X} \longrightarrow X \qquad [\gamma] \longmapsto \gamma(1) \in X$

↗ surjective

⇒ $\tilde{X}$ is the abstract construction of the universal cover

⇝ restrict to subgroups H of $\pi_1(X)$

⇝ $\tilde{X}_H$

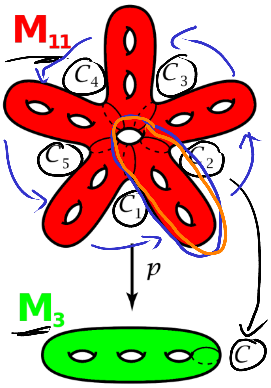
⇝ $\tilde{X}_H \xrightarrow{p_H} X \quad \forall\, H$ subgroups

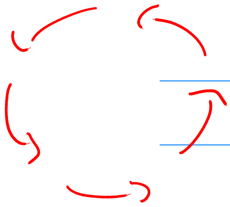
⇝ analyze how these vary as you vary H

⇝ Proof of

Theorem 1.38. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ and the set of subgroups of $\pi_1(X,x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ to the covering space $(\tilde{X},\tilde{x}_0)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p:\tilde{X}\to X$ and conjugacy classes of subgroups of $\pi_1(X,x_0)$.*

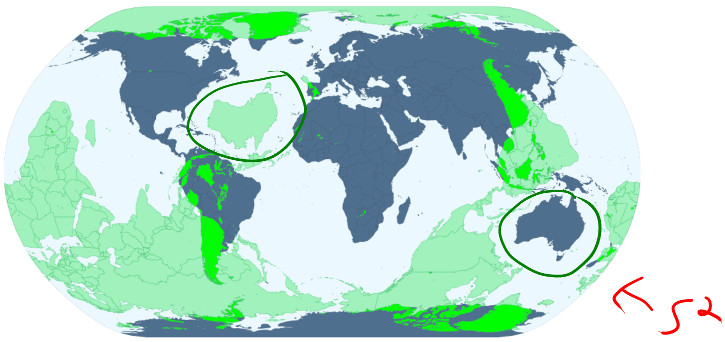
"Problem": Not explicit





- The surface M_{11} of genus 11 has a $G = \mathbb{Z}/5\mathbb{Z}$ symmetry Groups action
- Identifying along orbits gives $M_{11}/G \simeq M_3$ the surface of genus 3 Quotient
- M_{11} has a projection map to $M_{11}/G \simeq M_3$ Covering

⇝ should give us that group actions on spaces give i) Quotient spaces ii) Coverings



- S^2 has a $G = \mathbb{Z}/2\mathbb{Z}$ symmetry given by $x \mapsto -x$ Groups action
- Identifying along orbits gives $S^2/G \simeq \mathbb{R}P^2$ the real projective plane Quotient
- S^2 has a projection map to $S^2/G \simeq \mathbb{R}P^2$ Covering

⇝ This implies $\pi_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2\mathbb{Z}$

An action of a group G on a topological space X is a homomorphism

$$G \to \mathrm{Homeo}(X) = \{f : X \to X \mid f \text{ homeomorphism}\}$$

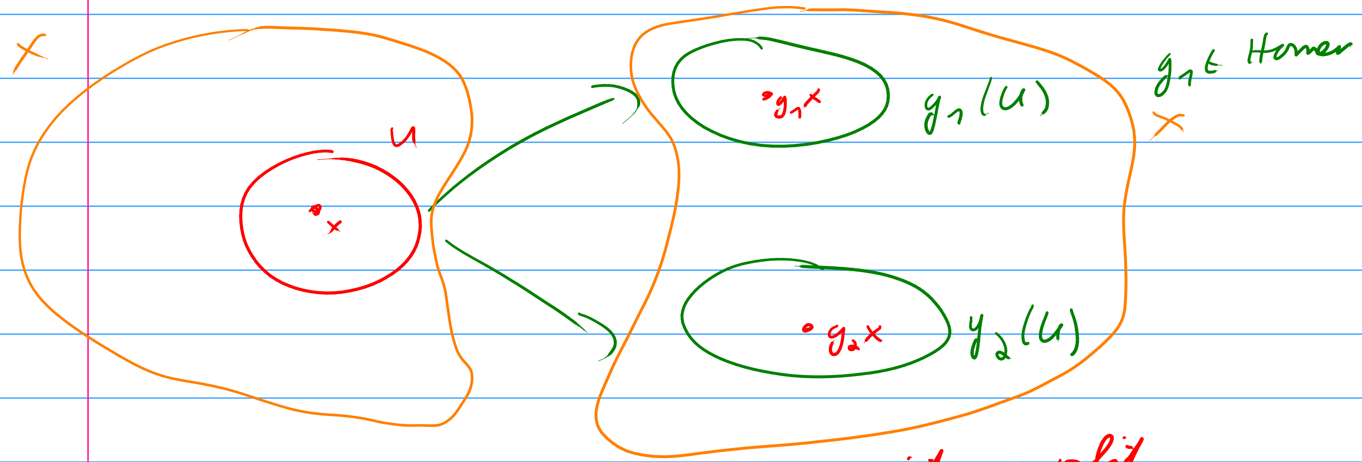
$G(X)$ in Hatcher

$G \curvearrowright X$

$g \mapsto \mathrm{Aut}(X)$ action of G on a set

Such an action is a covering action (a good action) if

$\forall x \in X \exists$ open neighborhood $U : g_1.U \cap g_2.U = \emptyset$ unless $g_1 = g_2$ *



Proposition

points are orbits

- The quotient of a covering action $p\colon X \to X/G$ is a covering
- If X is additionally path-connected and locally path-connected, then $G \cong \pi_1(X/G)/p_*(\pi_1(X))$

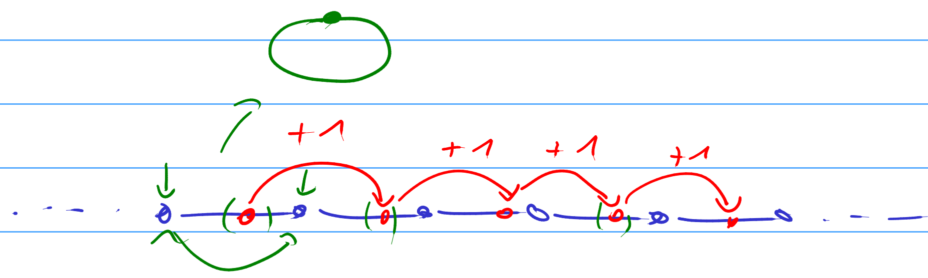
  $p_*(\pi_1(X))$ is normal in $\pi_1(X/G)$
- Special cases of good actions are Deck transformations: $f \in \mathrm{Homeo}(\tilde{X})$ with $p \circ f = p$ for $p\colon \tilde{X} \to X$

$$G \cong \pi_1(X/G)/p_*(\pi_1(X)) \qquad \pi_1(X) = 1$$

$$\Rightarrow G \cong \pi_1(X/G)$$

Form a topological space X/G whose points are orbits $\{g.x \mid g \in G\}$

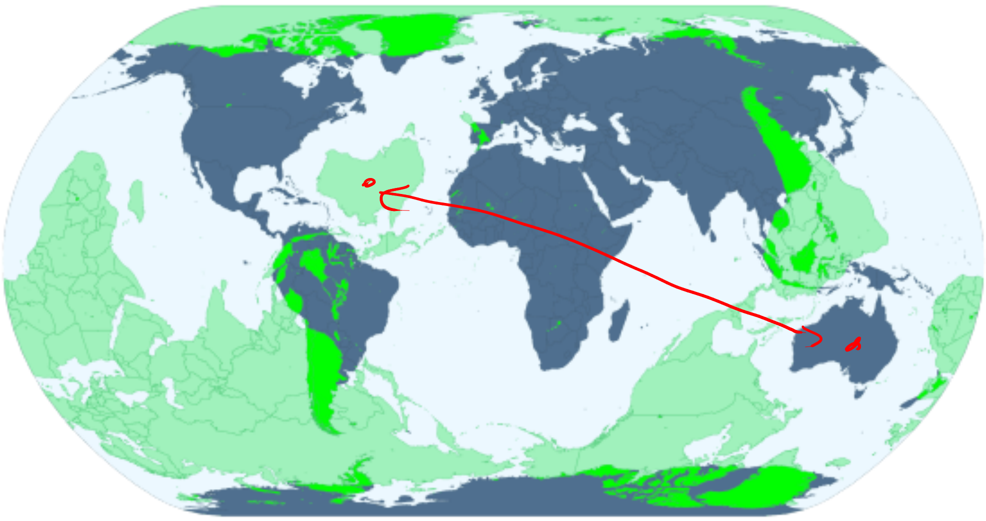
$G = \mathbb{Z} \curvearrowright \mathbb{R}$ translation



$\rightsquigarrow \mathbb{R}/\mathbb{Z} \cong S^1$

$$\Rightarrow G \simeq \pi_1(\underset{S^1}{\mathbb{R}/\mathbb{Z}}) / \underbrace{p_*(\pi_1(\mathbb{R}))}_{1} \simeq \pi_1(\mathbb{R}/\mathbb{Z})$$

$$\Rightarrow \pi_1(S^1) \simeq \mathbb{Z}$$

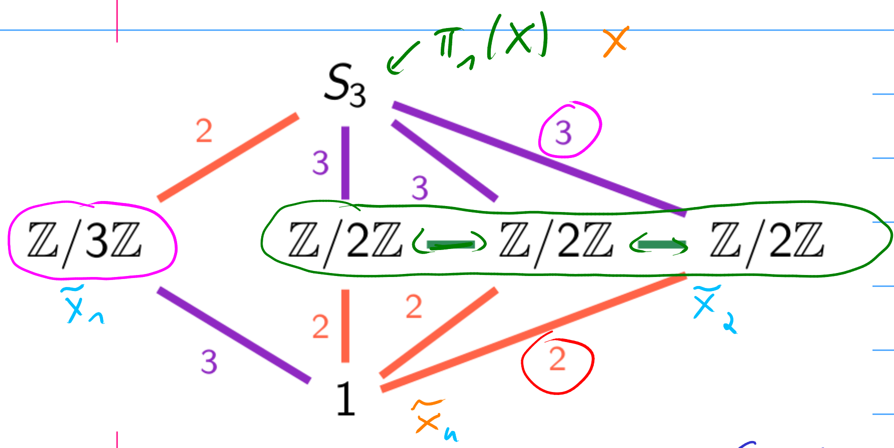


$S^2/(x \sim -x)$

$\cong S^2/\mathbb{Z}/2\mathbb{Z}$

$$\pi_1(S^2) \simeq 1 \Rightarrow \pi_1(\overbrace{S^2/\mathbb{Z}/2\mathbb{Z}}^{\mathbb{RP}^2}) \simeq \mathbb{Z}/2\mathbb{Z}$$

Construction of π_1 top-to-bottom

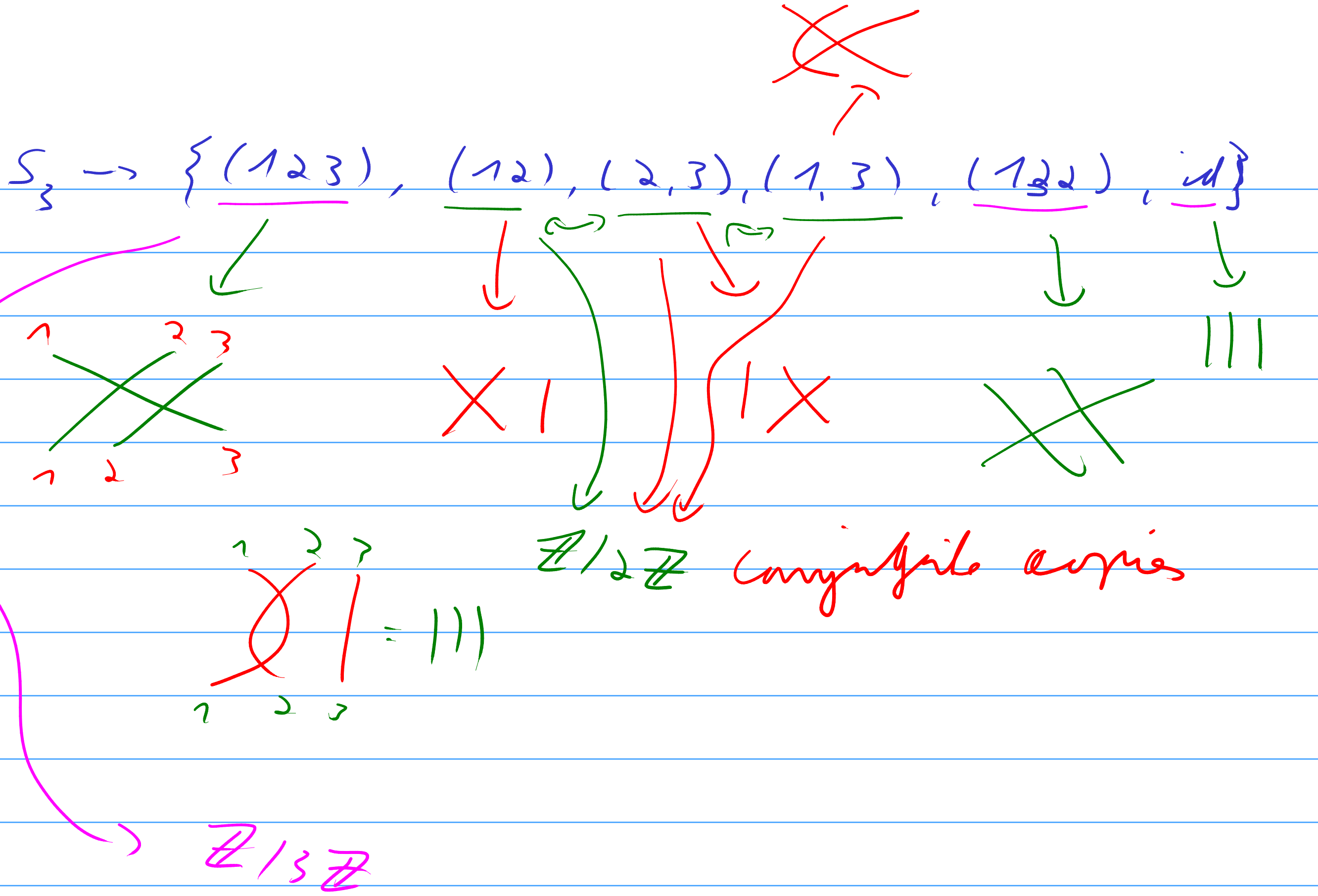


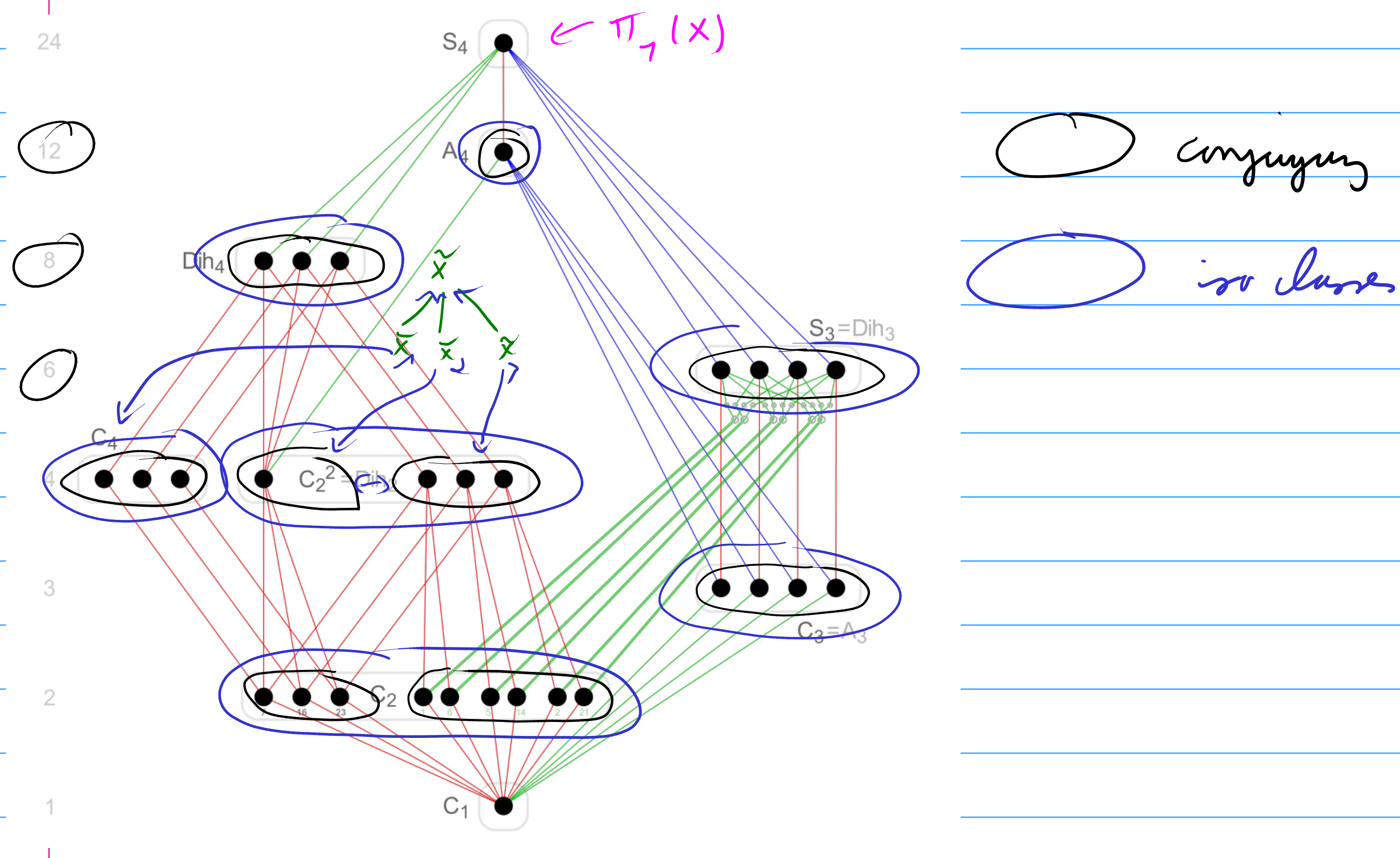
4 iso classes ignoring base points

6 with base points

group prop $\Longleftrightarrow$ Galois

Coverings


S3 → {(123), (12), (2,3), (1,3), (132), id}
Z/2Z conjugate copies
Z/3Z


← π1(X)
24
12
8
6
4
3
2
1
S4
A4
Dih4
C4
C2² = Dih2
S3 = Dih3
C3 = A3
C2
C1
conjugacy
iso classes

$G(\tilde{X})$

A cover $p\colon \tilde{X} \to X$ is normal if for all $x \in X$, and all $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$ there exists $\phi \in \mathrm{Homeo}(\tilde{X})$ with $\phi(\tilde{x}_1) = \tilde{x}_2$

Proposition 1.39. *Let $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ be a path-connected covering space of the path-connected, locally path-connected space X, and let H be the subgroup $p_*(\pi_1(\tilde{X},\tilde{x}_0)) \subset \pi_1(X,x_0)$. Then:*

(a) *This covering space is normal iff H is a normal subgroup of $\pi_1(X,x_0)$.*

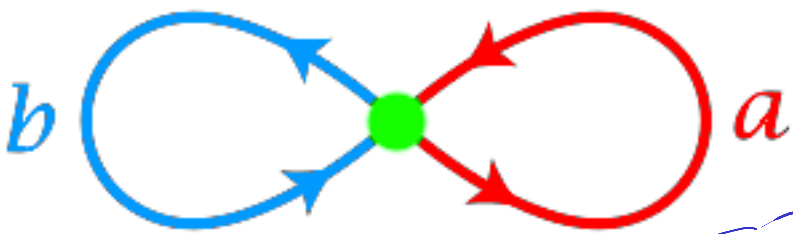
(b) *$G(\tilde{X})$ is isomorphic to the quotient $N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X,x_0)$.*

In particular, $G(\tilde{X})$ is isomorphic to $\pi_1(X,x_0)/H$ if $\tilde{X}$ is a normal covering. Hence for the universal cover $\tilde{X}\to X$ we have $G(\tilde{X}) \approx \pi_1(X)$.

~ Galois correspondence coverings <=> subgroups

- top-bottom approach to π_1

-> normal actions <=> normal subgroups

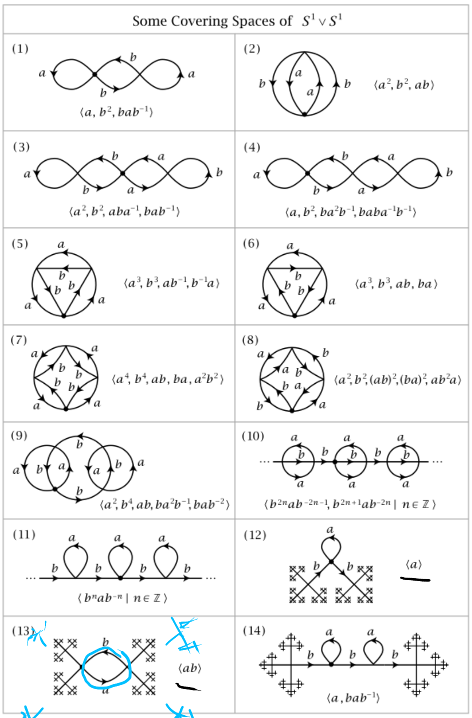


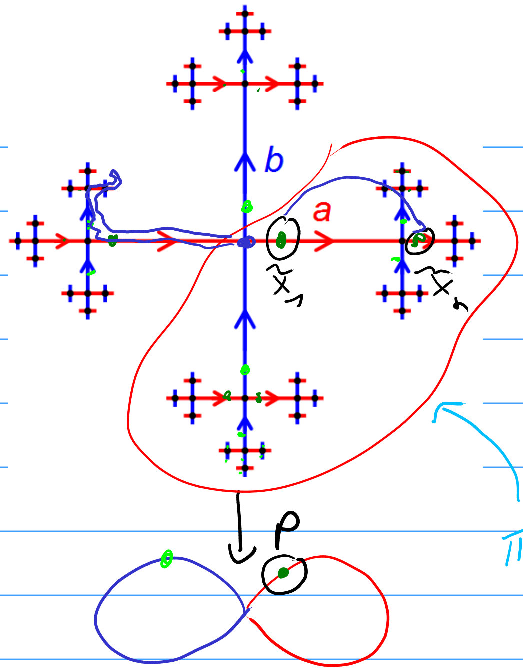
$S^1 \vee S^1$

$\pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z}$

a b

huge and has many subgroups



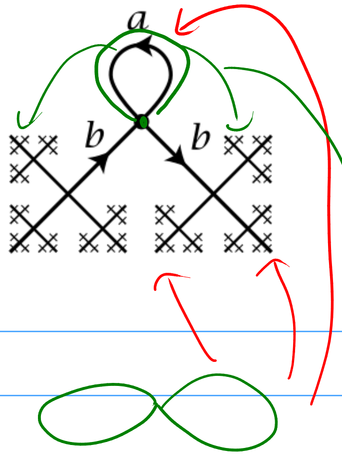


$1 * 1 \subset \dfrac{\mathbb{Z} * \mathbb{Z}}{F_2}$

Cayley graph of F_2

$\pi_1(\tilde{X}) = 1$

(12)

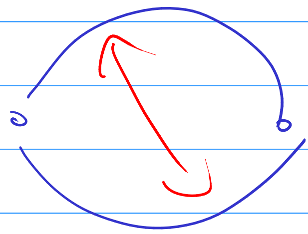


Not normal

$\langle a \rangle$

$\mathbb{Z} * 1 \subset \mathbb{Z} * \mathbb{Z}$

$\pi_1(\tilde{X}') \simeq \mathbb{Z}$



$\simeq \mathbb{RP}^2$

$\hookrightarrow \pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}/2\mathbb{Z}$