Groups and π_1

Plan $G \rightsquigarrow X$ such that $\pi_1(X) \simeq G$

explicit

Cayley complexes

2d cell complex

$G \rightsquigarrow$ Cayley graph

$\rightsquigarrow$ Cayley complex

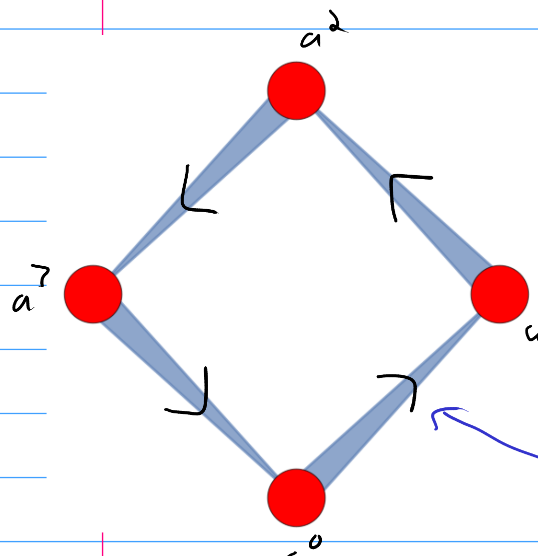
π_1

$K(G,1)$ space

Eilenberg-MacLane space

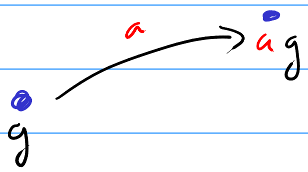
∞ d cell complexes

"essentially unique"

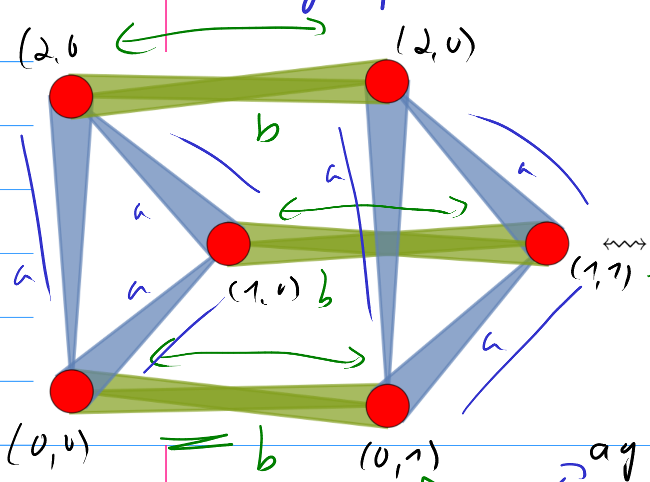


$\rightsquigarrow \mathbb{Z}/4\mathbb{Z}$

$= \langle a \mid a^4 = 1 \rangle$



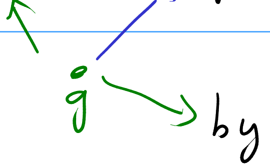
$\rightsquigarrow$ graph Γ associated to $\mathbb{Z}/4\mathbb{Z}$



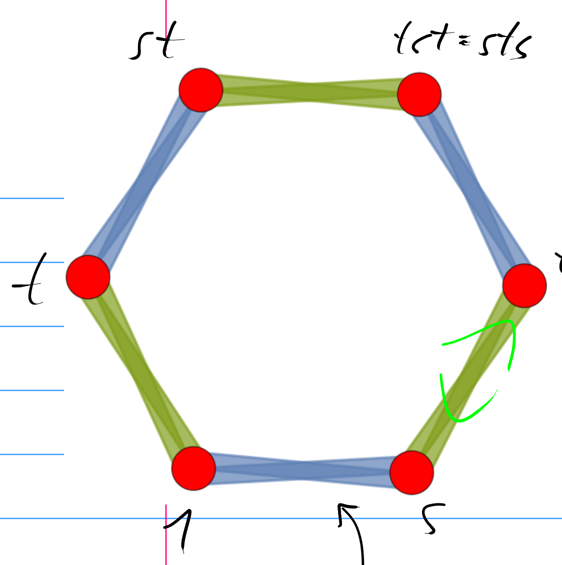
$\rightsquigarrow \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

↳ $\mathbb{Z}/3\mathbb{Z} \rightsquigarrow \langle a \mid a^3 = 1 \rangle$

$\mathbb{Z}/2\mathbb{Z} \rightsquigarrow \langle b \mid b^2 = 1 \rangle$



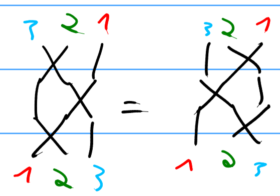
$\rightsquigarrow \Gamma_{\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$



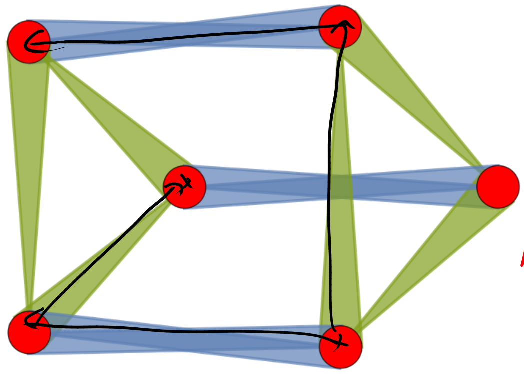
$\quad 1 \quad s \quad t$

$\rightsquigarrow S_3 = \langle (1,2), (2,3) \rangle$

$\times 1 \qquad 1 \times$

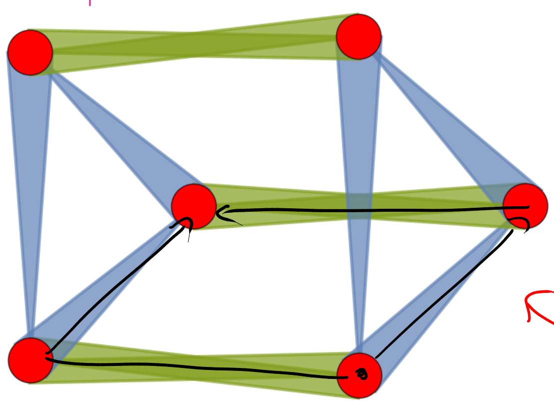


$\rightsquigarrow$ Cayley graph depends on the choice of $G = \langle \underbrace{S}_{\text{gens}} \mid \underbrace{R}_{\text{rels}} \rangle$



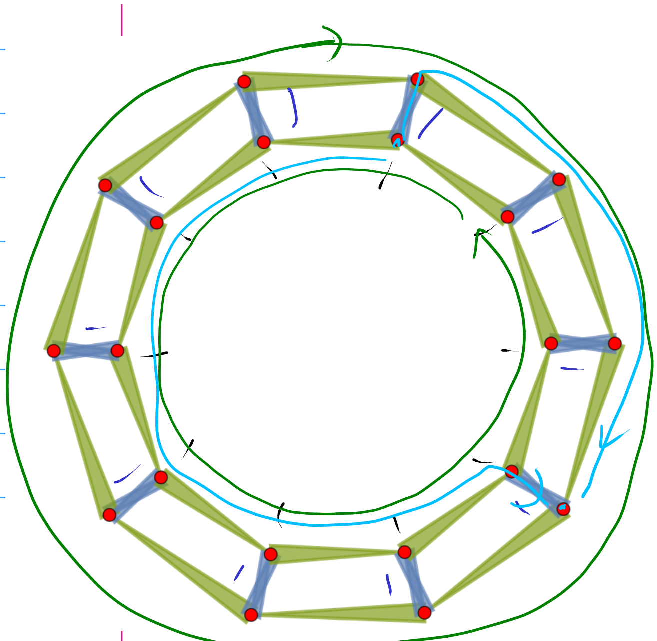
$\rightsquigarrow S_3 = \langle (1,2), (1,3,2) \rangle$

non-commutative



$\rightsquigarrow \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

commutative



20 element group

Dihedral group D_{10}

Symmetries of a 10-gon

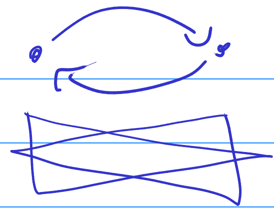
For a group $G = \langle S \rangle$ the Cayley graph $\Gamma = \Gamma(G, S)$ is constructed by:

(a) The vertex set of Γ is G

(b) Each $s \in S$ is assigned a color s

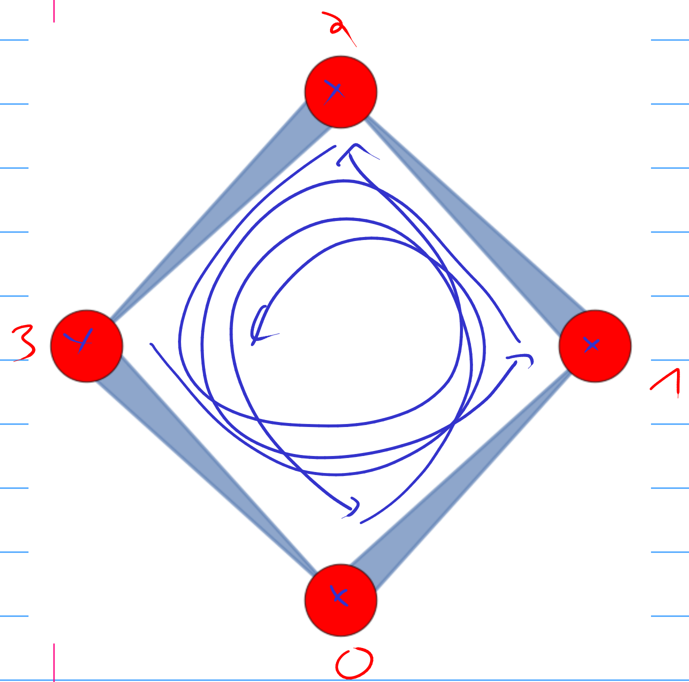
(c) Draw an edge of color s from g to gs

$g \to gs \to gss \overset{\shortparallel}{} g$

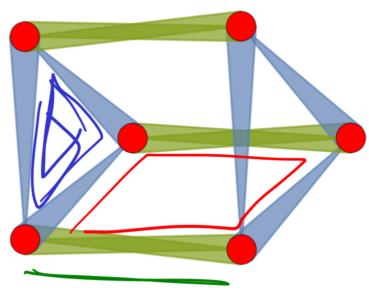


⇝ Where are the relations??

- Generators with $s = s^{-1}$ correspond to double edges ✓
- Cayley graphs are strongly connected ← follows because elements are invertible
- G is commutative if and only if two-step-walks commute **Commutative** ✓
- Closed walks are relations among words **Relations**



$\mathbb{Z}/4\mathbb{Z}$

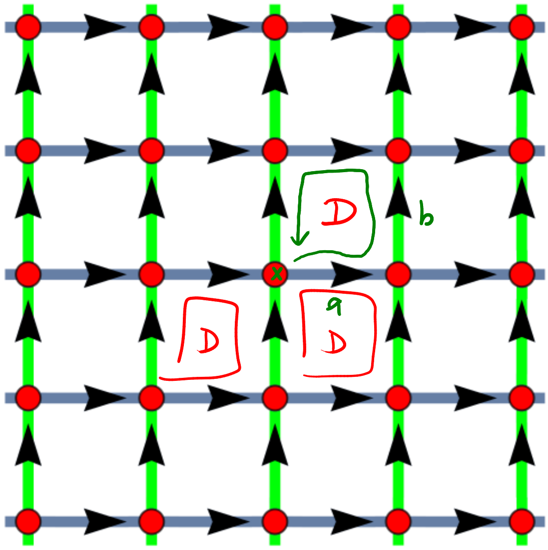


$\rightsquigarrow \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Relations are encoded in closed walks :)

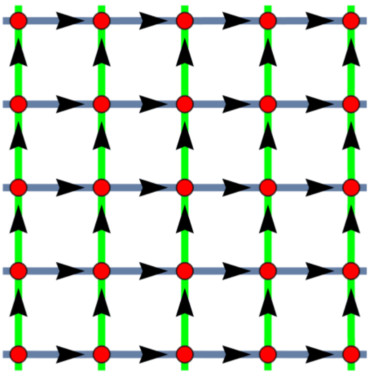
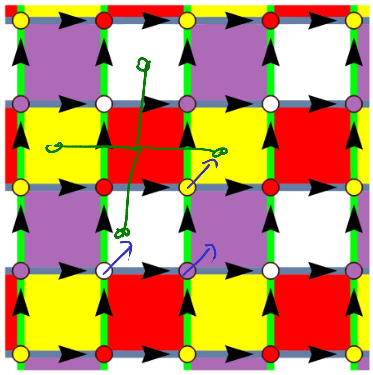
Problem: $\pi_1(\text{graph}) \simeq \ast_e \mathbb{Z}$ $\quad$ e in the complement of a spanning tree

Idea: Glue in discs D_2 for relations / closed walks.
This works :)

$\Gamma_{\mathbb{Z}\times\mathbb{Z}} =$ 

$aba^{-1}b^{-1}$

$\pi_1(\text{Torus}) \simeq \mathbb{Z}\times\mathbb{Z}$

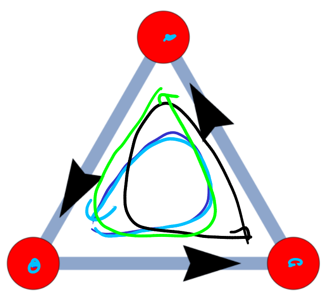
$\Gamma_{\mathbb{Z}\times\mathbb{Z}} =$  $\longrightarrow \mathbb{Z}\times\mathbb{Z} \circlearrowright \tilde{X}_{\mathbb{Z}\times\mathbb{Z}} \simeq$ 

$\mathbb{R}^2 \cong$

$aba^{-1}b^{-1}$

$\tilde{X}_{\mathbb{Z}\times\mathbb{Z}} / \mathbb{Z}\times\mathbb{Z} \simeq T$

Great: $\pi_1(T) \simeq \mathbb{Z}\times\mathbb{Z}$

$\mathbb{Z}/3\mathbb{Z} \cong \langle a \mid a^3 = 1\rangle \leftrightsquigarrow \Gamma_{\mathbb{Z}/3\mathbb{Z}} =$ 

$\pi_1(\Gamma_{\mathbb{Z}/3\mathbb{Z}}) \cong \mathbb{Z}$

$\pi_1(\Gamma) \simeq \mathbb{Z}$

$\tilde{X}_G$ is constructed by gluing discs for each $g \in G$ and $r \in R$

Cover of the Cayley complex ← $\langle S, R\rangle$

which $\tilde{X}_G$

Γ_G ← Cayley graph

↳ __ $= 1$

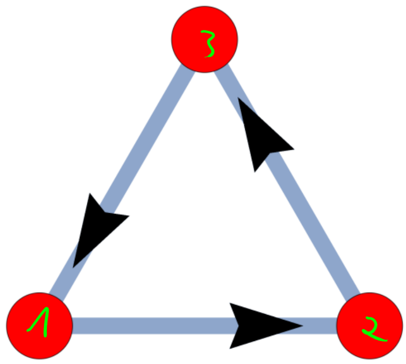
$X_G = \tilde{X}_G / G$

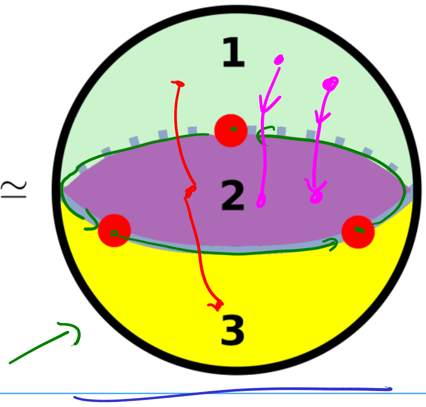
$aba^{-1}b^{-1} = 1$ ✓

$ab = ba$ ×

Cayley complex

G acts on $\tilde{X}_G$, so we obtain $X_G = \tilde{X}_G / G$

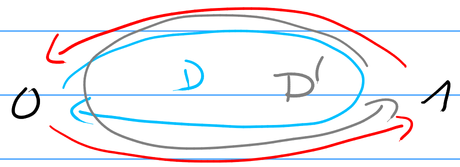
$\Gamma_{\mathbb{Z}/3\mathbb{Z}} =$ 

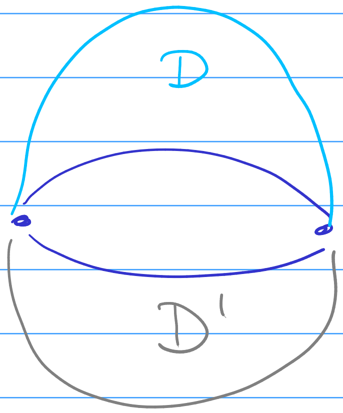
$\longrightarrow \mathbb{Z}/3\mathbb{Z} \circlearrowright \tilde{X}_{\mathbb{Z}/3\mathbb{Z}} \simeq$ 

$X_{\mathbb{Z}/3\mathbb{Z}} \simeq \tilde{X}/\mathbb{Z}/3\mathbb{Z}$ $\quad \pi_1(X_{\mathbb{Z}/3\mathbb{Z}}) \simeq \mathbb{Z}/3\mathbb{Z}$

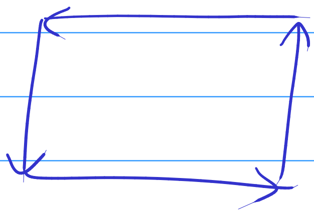
Example: $\mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 = 1 \rangle$

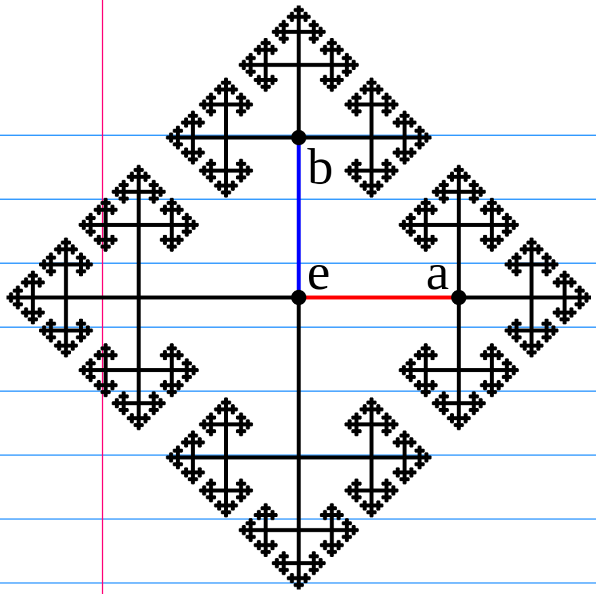
$= \{0, 1\}$

$\Gamma_{\mathbb{Z}/2\mathbb{Z}} =$ 

$\mathbb{Z}/2\mathbb{Z} \circlearrowright$ 

$= \tilde{X}_{\mathbb{Z}/2\mathbb{Z}}$

$X_{\mathbb{Z}/2\mathbb{Z}} =$  $\simeq \mathbb{RP}^2$



$= \Gamma_G$ $\quad G = \langle a, b \mid \emptyset \rangle$

Covering $\tilde{X}_G \to X_G$ and $\pi_1(\tilde{X}_G) \cong 1$ gives $\pi_1(X_G) \cong G$

Given a group G by generators–relations, *i.e.* $G \cong \langle S \mid R \rangle$, the Cayley complex $\tilde{X}_G$ of G is defined by:

cover of

(a) $\tilde{X}_G$ is 2-dimensional CW complex

(b) The 0-cell and 1-cells form the Cayley graph Γ_G ← 1-skeleton

(c) For each $g \in G$ and $r \in R$ there is a 2-cell $e_{g,r}$

(d) $e_{g,r}$ is glued to Γ_G starting at g and reading along r Discs

(e) X_G Cayley complex is $\tilde{X}_G / G$

2d space

Upshot: For a given $G \overset{\text{get}}{\rightsquigarrow}$ space X_G with $\pi_1(X_G) \cong G$

"Problem": The above crucially depends on $\langle S \mid R \rangle$

Question: Can we avoid choices and produce X with $\pi_1(X) \cong$ given G such that

- the homotopy type of X is fixed

$\pi_1(X) = G$ + ???

Wish list: - add a few assumptions $(*)$

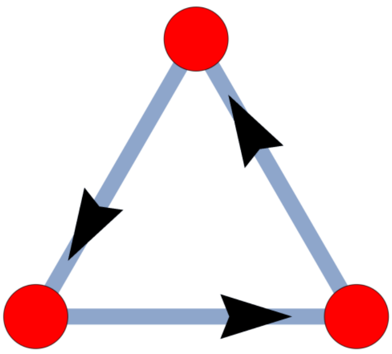
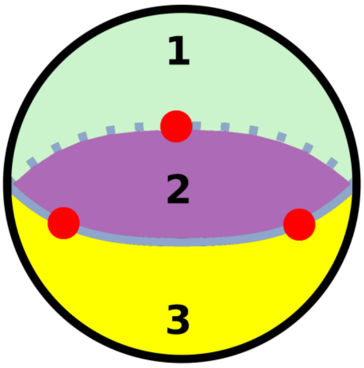
- Produce a space X
- Any other Y with $(*)$ satisfies $X \simeq Y$

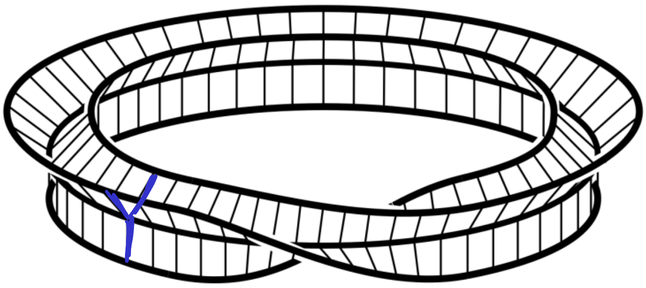
$\rightsquigarrow$ X is essentially uniquely associated to G

Problem: $\pi_1(X) \simeq \pi_1(Y) \not\Rightarrow X \simeq Y$

Turns out that fixing homotopy type of universal $\tilde{X}$ nails down X

- $\pi_1(\tilde{X}) \simeq 1$ always
- Fix the $\simeq$-equivalence of $\tilde{X}$
- Then determine X up to $\simeq$

$\Gamma_{\mathbb{Z}/3\mathbb{Z}} =$  $\longrightarrow \mathbb{Z}/3\mathbb{Z} \circlearrowright \tilde{X}_{\mathbb{Z}/3\mathbb{Z}} \simeq$ 

$X_{\mathbb{Z}/3\mathbb{Z}} \simeq$ ( plus glued disc along the boundary circle)

Y

---

x $\qquad \pi_1(X) \simeq G$

A path-connected space whose fundamental group is isomorphic to a given group G and which has a contractible universal covering space is called a **K(G,1) space**. The

$\tilde{X} \simeq \cdot$

wish!?

Eilenberg - MacLane spaces

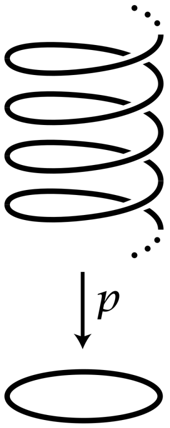
⇝ "Type of universal contraction"

Theorem 1B.8. *The homotopy type of a CW complex $K(G,1)$ is uniquely determined by G.*

↖ Uniqueness

Existence ?

Fixing a homotopy equivalence class of spaces with $\pi_1(\tilde{X}) \simeq 1$ determines spaces with $\pi_1(X) \simeq G$ up to homotopy



$\mathbb{R}$

S^1

$\mathbb{R} \simeq *$

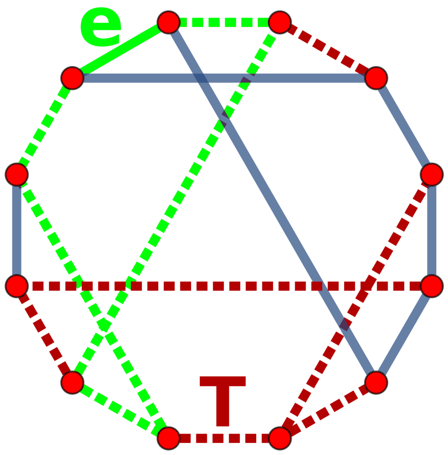
$\pi_1(\mathbb{R}) \simeq 1$

$\downarrow p$

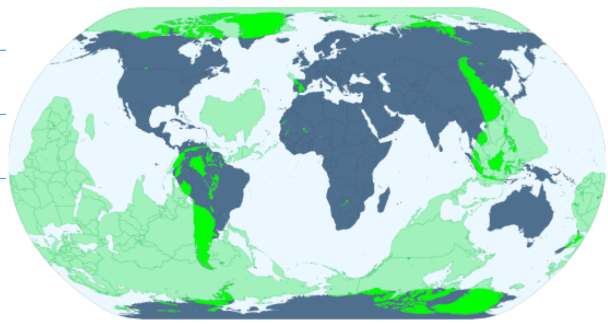
$\pi_1(S^1) \simeq \mathbb{Z}$

$\Rightarrow S^1 \simeq K(\mathbb{Z},1)$-space

Example 1B.1. S^1 is a $K(\mathbb{Z},1)$. More generally, a connected graph is a $K(G,1)$ with G a free group, since by the results of §1.A its universal cover is a tree, hence contractible.



not contractible

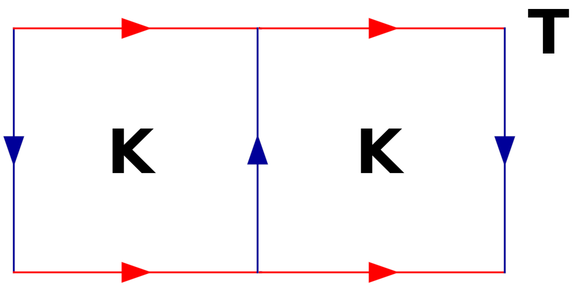


$\pi_1(S^2) \simeq 1$

$\downarrow$

$\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow \mathbb{RP}^2$ is not a $K(\mathbb{Z}/2\mathbb{Z},1)$-space



S^1
$\mathbb{R}^2$ $\quad \pi_1 \simeq 1$
$\downarrow$
T $\quad \pi_1 \simeq \mathbb{Z}\times\mathbb{Z}$
$\downarrow$
K $\quad \pi_1 \simeq (a,b \mid abab^{-1}=1)$
$\quad\quad\quad\quad\quad\quad \| $
$\quad\quad\quad\quad\quad\quad G_K$

$\rightsquigarrow$ $\circ$ is a $K(1,1)$-space
T is a $K(\mathbb{Z}\times\mathbb{Z},1)$-space
K is a $K(G_K,1)$-space

Example 1B.2. Closed surfaces with infinite π_1, in other words, closed surfaces other than S^2 and $\mathbb{R}P^2$, are $K(G,1)$'s. This will be shown in Example 1B.14 below. It also follows from the theorem in surface theory that the only simply-connected surfaces without boundary are S^2 and $\mathbb{R}^2$, so the universal cover of a closed surface with infinite fundamental group must be $\mathbb{R}^2$ since it is noncompact. Nonclosed surfaces deformation retract onto graphs, so such surfaces are $K(G,1)$'s with G free.

M $\quad |\pi_1(M)|=\infty \Rightarrow \mathbb{R}^2 \to M$ universal cover $\rightsquigarrow$ all of them are $K(G,1)$-spaces

Example 1B.3. The infinite-dimensional projective space $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2,1)$ since its universal cover is S^∞, which is contractible. To show the latter fact, a homotopy from the identity map of S^∞ to a constant map can be constructed in two stages as follows. First, define $f_t:\mathbb{R}^\infty\to\mathbb{R}^\infty$ by $f_t(x_1,x_2,\cdots) = (1-t)(x_1,x_2,\cdots)+t(0,x_1,x_2,\cdots)$. This takes nonzero vectors to nonzero vectors for all $t\in[0,1]$, so $f_t/|f_t|$ gives a homotopy from the identity map of S^∞ to the map $(x_1,x_2,\cdots)\mapsto(0,x_1,x_2,\cdots)$. Then a homotopy from this map to a constant map is given by $g_t/|g_t|$ where $g_t(x_1,x_2,\cdots) = (1-t)(0,x_1,x_2,\cdots)+t(1,0,0,\cdots)$.

$\mathbb{Z}/2\mathbb{Z}$

$\mathbb{R}P^\infty$ is the $K(\mathbb{Z}/2\mathbb{Z},1)$-space

Example 1B.4. Generalizing the preceding example, we can construct a $K(\mathbb{Z}_m,1)$ as an infinite-dimensional lens space $S^\infty/\mathbb{Z}_m$, where $\mathbb{Z}_m$ acts on S^∞, regarded as the unit sphere in $\mathbb{C}^\infty$, by scalar multiplication by m^{th} roots of unity, a generator of this action being the map $(z_1,z_2,\cdots)\mapsto e^{2\pi i/m}(z_1,z_2,\cdots)$. It is not hard to check that this is a covering space action.

$K(\mathbb{Z}/m\mathbb{Z},1)$-spaces are ∞-cell complex

Theorem: $K(G,1)$-space for G finite is a ∞-cell complex

Existence:

$(n+1)$-tuples $[g_0, \cdots, g_n]$ of elements of G

$(n+1)$-cells are given by $(n+1)$-tuples of elements of G

$G = \{0,1\} \simeq \mathbb{Z}/2\mathbb{Z}$

$3 \left\{ \begin{array}{l} (0,0,0) \\ (0,0,1) \\ (0,1,0) \\ (1,0,0) \\ (0,1,1) \\ (1,0,1) \\ (1,1,0) \\ (1,1,1) \end{array} \right.$

$(1,0)$, $(0,0)\times$, $(0,1)\times$

$\forall n$

$(1,1)$, $(1,1)$, $(1,1)$

$(n-1)$-simplices $[g_0, \cdots, \hat{g}_i, \cdots, g_n] \leftarrow \tilde{X}$

$n?$

Group G acts on it:

$0 \curvearrowright (1,0,1) = (1,0,1)$

$[gg_0, \cdots, gg_n]$ $1 \curvearrowright (1,0,1) = (0,1,0)$

$\rightsquigarrow G$ acts on $\tilde{X}$ $\rightsquigarrow$ $X \simeq \tilde{X}/G$

$\rightsquigarrow$ X is a $K(G,1)$-space

What needs to be done is that $\tilde{X} \simeq \bullet$

$BG = EG/G$, and BG is a $K(G,1)$

× $\tilde{×}$

Here is a list of important fundamental groups

- Spheres S^n

$$\pi_1(S^n) \cong \begin{cases} 1 & \text{if } n > 1 \\ \mathbb{Z} & \text{if } n = 1 \end{cases}$$

Problem!

- Torus T, real projective plane $\mathbb{R}P^2$ and Klein bottle K

$$\pi_1(T) \cong \mathbb{Z}^2, \quad \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}, \quad \pi_1(K) \cong \langle A_1, B_1, A_2, B_2 \mid A_1B_1A_1B_1^{-1} \rangle$$

- Orientable surfaces $M_{g,b}$ of genus $g > 0$ and b boundary points

$$\pi_1(M_{g,b}) \cong \langle A_1, B_1, \ldots, A_g, B_g, z_1, \ldots, z_b \mid [A_1, B_1] \cdot [A_g, B_g] = z_1 \ldots z_b \rangle$$

- Various topological groups $G/\mathbb{C}$ (all have commutative fundamental group)

G	$\mathbb{R}$	$\mathbb{Q}$	$\mathrm{GL}(n)$	$\mathrm{SL}(n)$	$\mathrm{SO}(2)$	$\mathrm{SO}(>2)$	$\mathrm{Sp}(n)$
π_1	1	1	$\mathbb{Z}$	1	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1

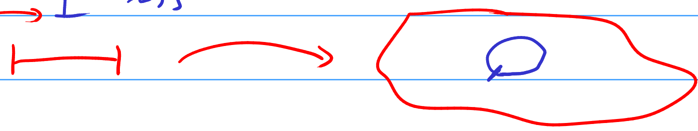
- Fundamental group of a graph Γ is $\pi_1(\Gamma) \cong *_e\mathbb{Z}$ where e runs over edges not contained in a spanning tree (discussed in a previous video)

⇝ We can not tell S^d apart from S^e Bad!

H_n does this

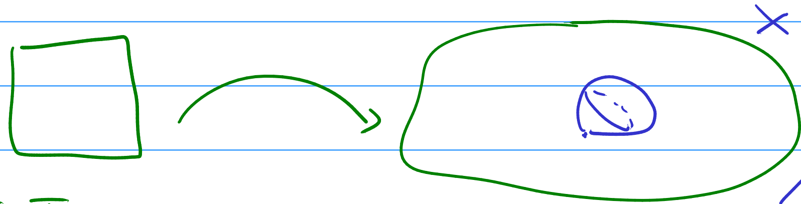
Idea: π_n n-dim "things"

1 dim → I ⇝ S^1



× / ≃

$H_n(S^d) \cong \begin{cases} \mathbb{Z} & n=0,d \\ 0 & \text{else} \end{cases}$



$I \times I$

× / ≃   ⇝ $\pi_2(X)$ Group

1930

Turns out π_n is: X + adjectives + π_n's ⇝ determines X

- very powerful
- almost impossible to compute $\pi_n(S^d)$ are not known!

⇝ Homology (H_n) is still very powerful but computable

⇝ replacement for π_n's