Lecture 8 : Axiomatic approach

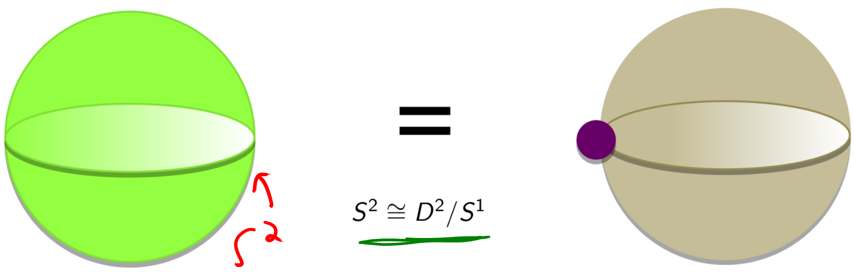
$\{v_i\}$ Basis V
$\{w_j\}$ Basis for W

Tensor product $\to$ $V \otimes W \rightsquigarrow$ has basis $v_i \otimes w_j$
$\to$ Universal property

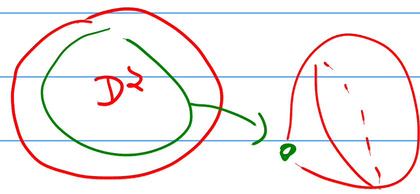
Existence $\Rightarrow$ Uniqueness

Existence

Homology $\to$ singular + cellular ... ~ 20
$\to$ Eilenberg-Steenrod axiom ~ 50
"Universal property of homology"



$S^2 \cong D^2/S^1$



The singular homology of the involved pieces

$H_i(S^n) \cong \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else} \end{cases}$ $\quad H_i(D^n) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{else} \end{cases}$ $\quad H_i(S^{n-1}) \cong \begin{cases} \mathbb{Z} & i = 0, n-1 \\ 0 & \text{else} \end{cases}$

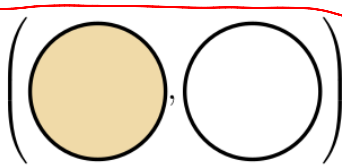
Quotient in Top $\xrightsquigarrow{H_*}$?? in algebra

What is the relation between those three homologies?

A topological pair (X, A): A space X and a subspace A (with subspace topology)

important

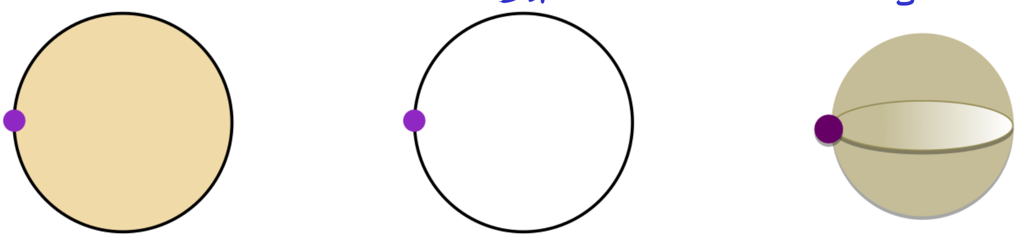
$(D^2, S^1) \leftrightsquigarrow$



Algebra	Topology
$W \hookrightarrow V \twoheadrightarrow V/W$	$C_*(A) \hookrightarrow C_*(X) \twoheadrightarrow C_*(X,A)$

incl quo incl quotient

D_2 S_1 S^2



$C_*(D^2)\colon \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$ $\quad C_*(S^1)\colon 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$ $\quad C_*(S^2)\colon \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}$

Not quite a quotient

$$\left(0 \xrightarrow{0} \boxed{\mathbb{Z} \xrightarrow{0} \mathbb{Z}}\right) \hookrightarrow \left(\mathbb{Z} \xrightarrow{1} \boxed{\mathbb{Z} \xrightarrow{0} \mathbb{Z}}\right)$$

$$\begin{tikzcd}
C_i \arrow[r, "\delta_C"] \arrow[d, "f"'] & C_{i-1} \arrow[r, "\delta_C"] \arrow[d, "f"] & C_{i-2} \arrow[d, "f"] \\
D_i \arrow[r, "\delta_D"'] & D_{i-1} \arrow[r, "\delta_D"'] & D_{i-2}
\end{tikzcd}$$

Map of complexes

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "id"] & \mathbb{Z} \arrow[d, "id"] \\
\mathbb{Z} \arrow[r, "1"'] & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z}
\end{tikzcd}$$

$\mathbb{Z} = \mathbb{Z}/0 \qquad \mathbb{Z}/\mathbb{Z} = 0 \qquad \mathbb{Z}/\mathbb{Z} = 0$

$\mathbb{Z} \longrightarrow 0 \longrightarrow 0$

$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}$?

$C_*(X,A) := C_*(X)/C_*(A)$

$H_*(X,A) =$ Homology of $C_*(X,A)$

- Relative homology $H_*(X, A)$ is the homology of $C_*(X, A) = C(X)/C(A)$
- $H_*(X, A) \ncong H_*(X)/H_*(A)$ in general
- $H_*(X/A) \ncong H_*(X, A)$ in general – but almost

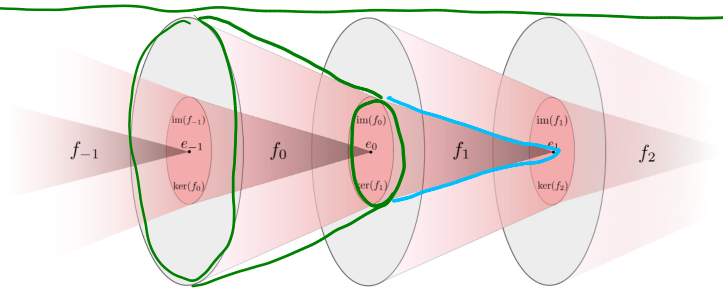
$B \rightsquigarrow (X, A)$ and that $X/A \cong B$ and X is contractible

$$(0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}) \hookrightarrow (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z}) \twoheadrightarrow (\mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \boxed{0})$$

S_1 D_2 "S_2"

$i-1$ i $i+1$ $i+2$ $i+3$

$$\cdots \xrightarrow{f_{-1}} G_{-1} \xrightarrow{f_0} G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots$$



exact at i: ingoing

$\ker f_{i+1} = \operatorname{Im} f_i$

outgoing

This is an exact sequence – kernels=images in any step

Exact sequences in topology "measure the failure of naive constructions to be true"

Outside zeros are often omitted, e.g.

$$0 \to (0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0) \xrightarrow{\to 0} \Longleftrightarrow (\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z})$$

$$\xrightarrow{0} \mathbb{Z} \xhookrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \quad \text{exact}$$

injective surjective

Short exact sequence (SES)

Given the following setup for a topological pair (X, A):

(a) $\iota\colon A \hookrightarrow X$ the inclusion of A into X

(b) π_* be induced by the projection $C_*(X) \twoheadrightarrow C_*(X, A)$

(c) $\partial\colon H_*(X, A) \to H_{*-1}(A)$ be the map that takes a relative cycle to its boundary

$H(X)/H(A)$

$\cong$

$H(X) \cong H(A) \oplus H(X, A)$

map to δ

then:

- There exists an exact sequence

$$\cdots \to H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{\pi_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

$X, A \quad A_{n-1} \quad X \quad X, A$

"incl" "main" "quo" failure

$H(X, A)$

$\rightsquigarrow H_n(X) \simeq H(A) \oplus H(X/A) \oplus$ "δ"

Short exact sequence (SES) is the appropriate replacement of $\oplus$:

$$A = \begin{array}{ccc} x & y & z \\ \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad B$$

$$\text{(SES)} \quad \mathbb{Q}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \mathbb{Q}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & -1 & 2 \end{pmatrix}} \mathbb{Q} \quad \text{and} \quad \mathbb{Q}^4 \cong \mathbb{Q}^3 \oplus \mathbb{Q}$$

$$\text{(SES)} \quad \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\text{qoutient}} \mathbb{Z}/2\mathbb{Z} \quad \text{but} \quad \mathbb{Z} \ncong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$\mathbb{Q}^3 \xrightarrow{A} \mathbb{Q}^4 \xrightarrow{B} \mathbb{Q}$$

$\ker A = 0$

$\operatorname{Im} B = \mathbb{Q}$

$\ker B = \operatorname{im} A$

$= \langle x, y, z \rangle$

$\mathbb{Q}^4 \simeq \mathbb{Q}^3 \oplus \mathbb{Q}$

$W \hookrightarrow V \twoheadrightarrow V/W$

$V/W \oplus W \simeq V$

Replacement for $\oplus$ is a (SES)

A (long) exact sequence is almost like a long sequence of $\oplus$:

$$\left(\cdots \to H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{\pi_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots\right) \text{“}\Rightarrow\text{”} H_n(X) \cong H_n(A) \oplus H_n(X, A) \oplus \text{“}\partial \text{ is not zero”}$$

- $A \neq \emptyset$ closed subspace+deformation retract of some neighborhood in X, then

$$\tilde{H}_*(X/A) \cong H_*(X,A)$$

The tilde means "get rid of the zero homology" (reduced homology)

$\tilde{H}_*$ reduced homology

$\tilde{H}_i \simeq H_i$ unless $i=0$

$H_0(\text{point}) \simeq \mathbb{Z}$

$\tilde{H}_0(\text{point}) \simeq 0$

$P(X) = t\,(\ldots)$

$\tilde{P}(X) = (\ldots)$

$$\begin{array}{ccccccccc}
\xrightarrow{\delta_2} & C_1 & \xrightarrow{\delta_1} & C_0 & \xrightarrow{\delta_0} & 0 \\
 & \| & & \| & & \\
\xrightarrow{\delta_2} & \tilde{C}_1 & \xrightarrow{\delta_1} & \tilde{C}_0 & \xrightarrow{\tilde{\delta}_0} & 0
\end{array}$$

$\delta_0(\sum n_i \sigma_i) = \sum n_i$

$$\underset{A}{H_2(S^1)} \xrightarrow{\iota_*} \underset{X}{H_2(D^2)} \xrightarrow{\pi_*} \underset{X,A}{H_2(D^2,S^1)} \xrightarrow{\partial} \underset{A}{H_1(S^1)} \xrightarrow{\iota_*} \underset{X}{H_1(D^2)}$$

$$\to \underset{X,A}{H_1(D^2,S^1)} \xrightarrow{\partial} \underset{A}{H_0(S^1)} \xrightarrow{\iota_*} \underset{X}{H_0(D^2)} \xrightarrow{\pi_*} \underset{X,A}{H_0(D^2,S^1)} \xrightarrow{0} 0$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} 0$$

$$\to 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} 0$$

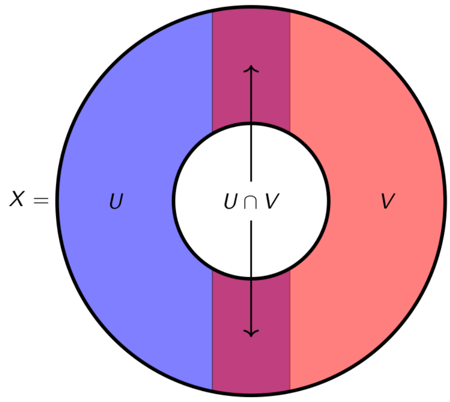
D^2 $\qquad (\underset{X}{D^2}, \underset{A}{S^1})$

$$0 \longrightarrow \mathbb{Z} \quad \overset{?}{\longrightarrow} \quad \mathbb{Z} \longrightarrow 0$$

$\Rightarrow \simeq$

$H_*(D^2, S^1) = 0 + t\cdot 0 + t^2$

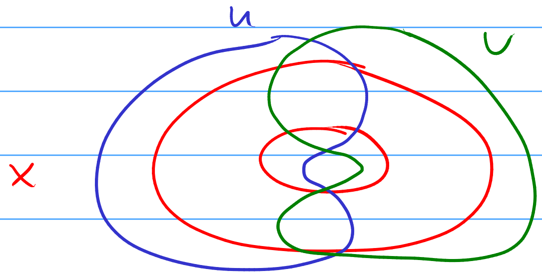
$H_*(S^2) = 1 + t^2.$

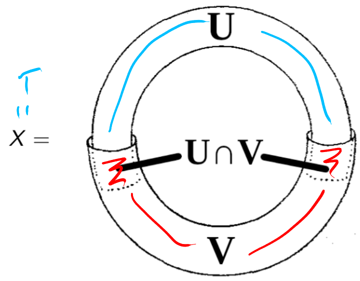


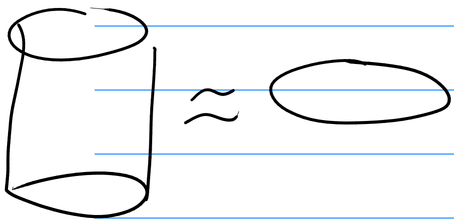
Mayer-Vietoris sequence

"Seifert van Kampen"

If we know $H_*(U), H_*(V), H_*(U \cap V)$, shouldn't we be able to compute $H_*(X)$?
Cutting into pieces





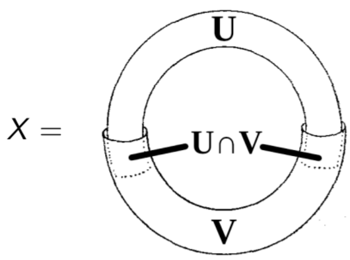


$U, V \rightsquigarrow$ cylinder, $U \cap V \rightsquigarrow$ two cylinders

- The Hilbert–Poincaré polynomials are

$$P(X) = 1 + 2t + t^2, \quad P(U) = P(V) = 1 + t, \quad P(U \cap V) = 2 + 2t$$

- Question Is there any relation between these?



- We get inequations on Hilbert–Poincaré polynomials:

$$1 + 2t + t^2 \leq (1 + t) + (1 + t) + t \cdot (2 + 2t)$$

$$P(X) \leq P(U) + P(V) + t \cdot P(U \cap V)$$

$$(1 + t) + (1 + t) \leq (1 + 2t + t^2) + (2 + 2t)$$

$$P(U) + P(V) \leq P(X) + P(U \cap V)$$

$$t \cdot (2 + 2t) \leq t \cdot ((1 + t) + (1 + t)) + (1 + 2t + t^2)$$

$$t \cdot P(U \cap V) \leq t \cdot (P(U) + P(V)) + P(X)$$

- The Mayer–Vietoris sequences tells you what to do to get equalities

For X any topological space with subspaces U, V whose interior cover X, we have an exact sequence

$$\cdots \xrightarrow{\partial_*} H_n(U \cap V) \xrightarrow{(i_*, j_*)} \overbrace{H_n(U) \oplus H_n(V)}^{H_n(U \cup V)} \xrightarrow{k_* - l_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(U \cap V) \to \cdots$$

$$\cdots \to H_1(X) \xrightarrow{\partial_*} H_0(U \cap V) \xrightarrow{(i_*, j_*)} H_0(U) \oplus H_0(V) \xrightarrow{k_* - l_*} H_0(X) \to 0$$

This needs a choice of order for U, V

$(U \cap V \quad U \oplus V \quad X) \quad (U \cap V \quad U \oplus V \quad X)$

$H(U \cup V) \simeq H(U \cap V) \oplus H(X) \oplus$ "δ_*"

$P(U) + P(V) = P(U \cup V) \leq P(U \cap V) + P(X)$

$H_n(X) \simeq H_n(U \cup V) \oplus H_{n-1}(U \cap V) \oplus$ "–"

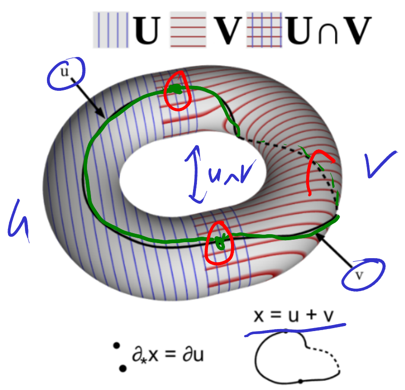
$P(X) \leq P(U \cup V) + tP(U \cap V)$

- Here we use the inclusions $i, j\colon U \cap V \hookrightarrow U, V$ and $k, l\colon U, V \hookrightarrow X$
- The boundary map ∂_* lowers the degree
- The three ways to cut out three bits give the inequations, e.g.

$$H_*(U) \oplus H_*(V) \to H_*(X) \to H_{*-1}(U \cap V)$$

$$P(X) \leq P(U) + P(V) + t \cdot P(U \cap V)$$

Boundaries $\partial_n\colon H_n(X) \to H_{n-1}(U \cap V)$



- A cycle x in $H_n(X)$ is sum of two chains u, v in U, V
- $\partial_n(x) = \partial_n(u + v) = 0 \Rightarrow \partial_n(u) = -\partial_n(v)$
- Both, $\partial_n(u), \partial_n(v)$ lie in $H_{n-1}(U \cap V)$
- Define $\partial_n(x) = \partial_n(u) \in H_{n-1}(U \cap V)$

$H_n(X) \to H_{n-1}(U \cap V)$

$\delta_n(x) = \delta_n(u)$

- $\mathbb{Z} \to R, 1 \mapsto 1$ gives a way to interpret integers as elements of any ring R
- Formally this can be encoded using $\_ \otimes_{\mathbb{Z}} R$
- We get $C_*(X,R) = C_*(X) \otimes_{\mathbb{Z}} R$ chain complexes with coefficients
- Some numbers will become invertible or zero (divisors) in $C_*(X,R)$

$\mathbb{Z} \to \mathbb{Q}$ $\quad 2 \rightsquigarrow$ invert

$\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ $\quad 2 \rightsquigarrow 0$

$$\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$
$$1 \longmapsto 1$$
$$n \longmapsto 0$$

$$1200 \longmapsto 1200 + n\mathbb{Z}$$

$$\dots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \underline{\mathbb{Z}} = C_*(\mathbb{R}P^\infty)$$

$$\dots \xrightarrow{0} \mathbb{Q} \xrightarrow{\cdot 2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{\cdot 2} \mathbb{Q} \xrightarrow{0} \underline{\mathbb{Q}} = C_*(\mathbb{R}P^\infty) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\dots \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\overset{0}{\cdot 2}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\overset{0}{\cdot 2}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \underline{\mathbb{Z}/2\mathbb{Z}} = C_*(\mathbb{R}P^\infty) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

$$H_n(\mathbb{R}P^\infty) \cong \begin{cases} \mathbb{Z} \text{ if } n = 0 \\ \mathbb{Z}/2\mathbb{Z} \text{ if } n \text{ is odd} \\ 0 \text{ else} \end{cases}$$

$$H_n(\mathbb{R}P^\infty, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} \text{ if } n = 0 \\ 0 \text{ else} \end{cases} \cong H_n(\mathbb{R}P^\infty) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{Same}$$

Universal coefficients theorem

UCT

$$H_n(\mathbb{R}P^\infty, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_n(\mathbb{R}P^\infty) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \text{ if } n = 0 \text{ or } n \text{ odd} \\ 0 \text{ else} \end{cases} \quad \text{Different}$$

- We get $H_*(X,R) =$homology of $C_*(X,R)$ homology with coefficients
- We could also naively change coefficients $H_*(X) \otimes_{\mathbb{Z}} R$
- The UCT measures the difference between $H_*(X,R)$ and $H_*(X) \otimes_{\mathbb{Z}} R$

naive

good

For any $\mathbb{Z}$-module R singular homology satisfies

- There exists an exact sequence

$$0 \to (H_n(X) \otimes_{\mathbb{Z}} R) \to H_n(X, R) \to \mathrm{Tor}(H_{n-1}(X), R) \to 0$$

- This sequence splits (not naturally)
- We have a direct sum decomposition

simple

$$H_n(X, R) \cong \underbrace{(H_n(X) \otimes_{\mathbb{Z}} R)}_{\text{naive}} \oplus \underbrace{\mathrm{Tor}(H_{n-1}(X), R)}_{\text{Tor}}$$

- $\mathbb{Z}$ is hence the "universal" coefficient group
- $\mathrm{Tor}(H_{n-1}(X), R)$ measures how far $\_ \otimes_{\mathbb{Z}} \_$ is from being exact
- This is a statement "in algebra" and holds more generally

$\mathrm{Tor}(A, B)$ is the homology of any free resolution of A tensored with B

Tor measures the failure of $\_ \otimes_{\mathbb{Z}} \_$ being exact:

$$0 \to B \to C \to D \to 0 \text{ exact} \Rightarrow$$
$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to A \otimes_{\mathbb{Z}} B \to A \otimes_{\mathbb{Z}} C \to A \otimes_{\mathbb{Z}} D \to 0 \text{ exact}$$

Basic tools for computing Tor:

- $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$ Commutative
- $\mathrm{Tor}(\bigoplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$ Additive
- $\mathrm{Tor}(A, B) \cong 0$ if A or B is torsionfree Often trivial, *e.g.* $\mathrm{Tor}(\mathbb{Q}, B) \cong 0$
- $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, B) \cong \ker(B \xrightarrow{\cdot n} B)$ Torsion, *e.g.* $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$

$$H_n(X, \mathbb{Q}) \simeq (H_n(X) \otimes \mathbb{Q}) \oplus 0$$

$\mathbb{R}$ $\mathbb{R}$

$\mathbb{C}$ $\mathbb{C}$

$$H(X\times Y) \overset{?}{\rightsquigarrow} H(X)\otimes H(Y)$$

Let X, Y be any topological spaces, and R a PID

- There are short (non-naturally) splitting exact sequences

error

$$\bigoplus_{p+q=n} H_p(X,R)\otimes_R H_q(Y,R) \to H_n(X\times Y,R) \to \bigoplus_{p+q=n-1} \mathrm{Tor}^R(H_p(X,R),H_q(Y,R))$$

- There are isomorphism of $\mathbb{Q}$-vector spaces

$$H_*(X,\mathbb{Q})\otimes_{\mathbb{Q}} H_*(Y,\mathbb{Q}) \cong H_*(X\times Y,\mathbb{Q})$$

No error terms

- In particular, $P(X\times Y) = P(X)P(Y)$

Product of spaces $\rightsquigarrow$ Product of homologie $/\mathbb{Q}$

$$H_*(T,\mathbb{Q}) = H_*(S^1,\mathbb{Q})\otimes H_*(S^1,\mathbb{Q})$$

$$\mathbb{Q}\otimes_{\mathbb{Q}}\mathbb{Q}\simeq\mathbb{Q} \qquad \mathbb{Q}\oplus t\mathbb{Q} \qquad \mathbb{Q}\oplus t\mathbb{Q}$$

$$\simeq \mathbb{Q}\oplus t\mathbb{Q}^2 \oplus t^2\mathbb{Q}$$

$$P(T) = 1 + 2t + t$$

$$P(S^1)P(S^1) = (1+t)^2 \,''$$

Axiom

A homology theory H_* satisfying the dimension axiom is a functor $H_*\colon \mathrm{Top}^2 \to \mathbb{Z}\mathrm{mod}$ from pairs of topological spaces to $\mathbb{Z}$-modules together with boundary maps $\partial = \partial_n(X, A)\colon H_n(X, A) \to H_{n-1}(A, \emptyset) = H_{n-1}(A)$ satisfying:

- Homotopic maps induce the same map in homology **Homotopy invariance**
- If (X, A) is a pair and $U \subset A$ such that its closure is contained in the interior of A, then the inclusion

$$\iota\colon (X \setminus U, A \setminus U) \to (X, U)$$

  induces an isomorphism in homology **Excision**

$X = (X, \emptyset)$

- Each (X, A) induces a long exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

  via the inclusions $i\colon A \hookrightarrow X$ and $j\colon X \hookrightarrow (X, A)$ **Exactness**

$H(U \cup V)$
$\simeq H(U) \oplus H(V)$

- Direct sums $\bigoplus_i H(X_i)$ correspond to disjoint unions $\coprod_i X_i$: they are isomorphic by the inclusions $(\iota_i)_*$ **$\oplus \leftrightsquigarrow \coprod$**

$\oplus \leftrightsquigarrow \coprod$

- $H_n(\text{point}) = 0$ for all $n > 0$, and $H_0(\text{point}) = \mathbb{Z}$ **Dimension axiom**

Uniqueness

Theorem. Let X be a cell complex. Then up to equivalence, there is only one homology theory satisfying the dimension axiom.

Singular + cellular + simplicial $\rightsquigarrow$ all the same

- Singular homology is a homology theory satisfying the dimension axiom
  Existence
- Singular homology is up to equivalence the only such theory (for all reasonable input spaces)
  Uniqueness
- This implies that

  $$\text{singular} = \text{simplicial} = \text{cellular}$$

  (for all reasonable input spaces)
- Mayer–Vietoris and co. follow from the abstract theory.
- One can compute *e.g.* the homology of spheres from the axioms alone and thus, proof theorems such as Brouwer's fixed point theorem and the hairy ball theorem
  No explicit arguments needed