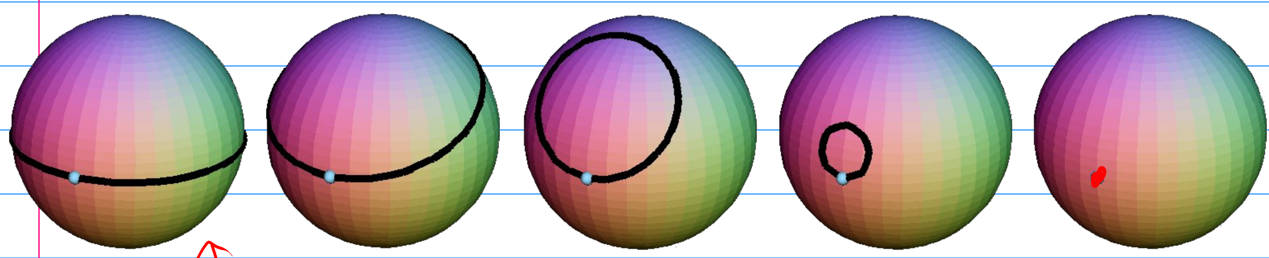
Slogan. Generators of $H_k(X)$ $\leftrightsquigarrow$ k-dimensional subspaces up to homotopy

$\hookrightarrow \underbrace{\mathbb{Z}^{\oplus k}}_{\text{free}} \oplus \underbrace{\mathbb{Z}/p^k\mathbb{Z} \oplus \dots}_{\text{torsion}}$



S^2

$H_1(S^2) \leftrightsquigarrow$ 1-dim submanifolds$/\simeq$

$\| \; 0$

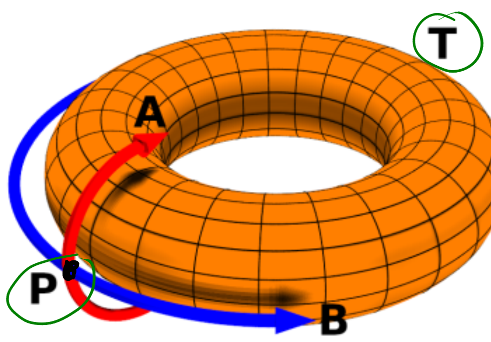
$H_0(X) \leftrightsquigarrow$ 0-dim subspaces$/\simeq$ $\leftrightsquigarrow$ connected components

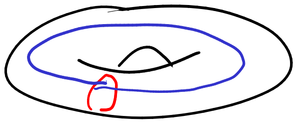
$H_2(S^2) \leftrightsquigarrow$ 2-dim submanifolds $\leftrightsquigarrow$ S^2 itself

$H_*\Bigg($  $\Bigg) \cong \mathbb{Z} \oplus t^2\mathbb{Z}$

dim 0 $\to t^0$ dim 2

$[P] \leftrightsquigarrow \mathbb{Z}$, $[S^2] \leftrightsquigarrow t^2\mathbb{Z}$

H_* () $\cong \mathbb{Z} \oplus t\mathbb{Z}^{\oplus 2} \oplus t^2\mathbb{Z}$



$\binom{2}{1}$

$[P] \leftrightsquigarrow \mathbb{Z}, [A], [B] \leftrightsquigarrow t\mathbb{Z}^{\oplus 2}, [T] \leftrightsquigarrow t^2\mathbb{Z}$

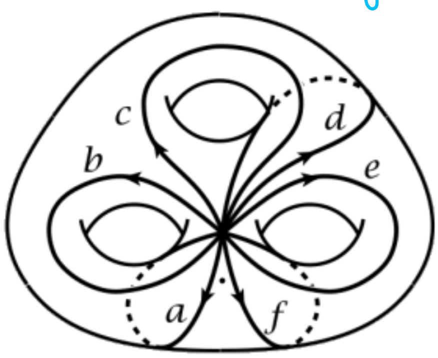
$$H_*(T^n) \cong \bigoplus_k t^k\mathbb{Z}^{\oplus \binom{n}{k}}$$

$T^n = \underbrace{S^1 \times \dots \times S^1}_{n}$

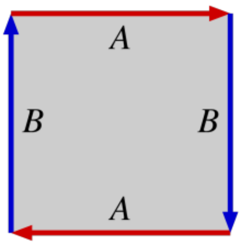
$[P] \leftrightsquigarrow \mathbb{Z}^{\oplus \binom{n}{0}}$, $\binom{n}{1}$ copies of $S^1 \times S^1 \leftrightsquigarrow t\mathbb{Z}^{\oplus \binom{n}{1}}, \dots, [T^n] \leftrightsquigarrow t^n\mathbb{Z}^{\oplus \binom{n}{n}}$

3-tori $[P]$, $[A],[B],[C]$, $\to 3\ T^2$'s, $[T^3]$

$g=3$

H_* () $\cong \mathbb{Z} \oplus t\mathbb{Z}^{\oplus 2g} \oplus t^2\mathbb{Z}$

$[P] \leftrightsquigarrow \mathbb{Z}, [A_1], [B_1], \dots, [A_g], [B_g] \leftrightsquigarrow t\mathbb{Z}^{\oplus 2g}, [M_{g,0}] \leftrightsquigarrow t^2\mathbb{Z}$

H_* () $\cong \mathbb{Z} \oplus t\mathbb{Z}/2\mathbb{Z} \oplus t^2 0$

$(x_1 : x_0)$

$\hookrightarrow \mathbb{R}P^1 \subset \mathbb{R}P^2$

$[P] \leftrightsquigarrow \mathbb{Z}, [\mathbb{R}P^1] \leftrightsquigarrow t\mathbb{Z}/2\mathbb{Z}, [\mathbb{R}P^2] \leftrightsquigarrow t^2 0??$

$$H_*(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) \cong \underline{\mathbb{Z}/2\mathbb{Z} \oplus t\mathbb{Z}/2\mathbb{Z} \oplus t^2\mathbb{Z}/2\mathbb{Z} \oplus ... \oplus t^n\mathbb{Z}/2\mathbb{Z}}$$

$$\underline{[P] \leftrightsquigarrow \mathbb{Z}/2\mathbb{Z}, [\mathbb{R}P^1] \leftrightsquigarrow t\mathbb{Z}/2\mathbb{Z}, [\mathbb{R}P^2] \leftrightsquigarrow t^2\mathbb{Z}/2\mathbb{Z}, ..., [\mathbb{R}P^n] \leftrightsquigarrow t^n\mathbb{Z}/2\mathbb{Z}}$$

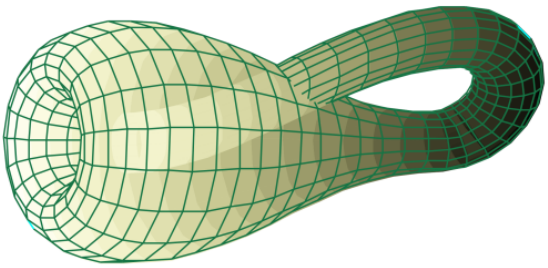
$C_*(\mathbb{R}P^n)$

$$\begin{tikzcd}
\cdots \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, shift right] & \mathbb{Z} \arrow[r, "2"] \arrow[d, shift right] & \mathbb{Z} \arrow[r, "0"] \arrow[d, shift right] & \mathbb{Z} \arrow[r] \arrow[d, shift right] & 0 \\
\cdots 0 \cdots & \mathbb{Z}/2\mathbb{Z} \arrow[u, shift right, "\mathbb{R}P^3"'] & 0 \arrow[u, shift right, "\mathbb{R}P^2"'] & \mathbb{Z}/2\mathbb{Z} \arrow[u, shift right, "\mathbb{R}P^1"'] & \mathbb{Z} \arrow[u, shift right, "P"'] &
\end{tikzcd}$$

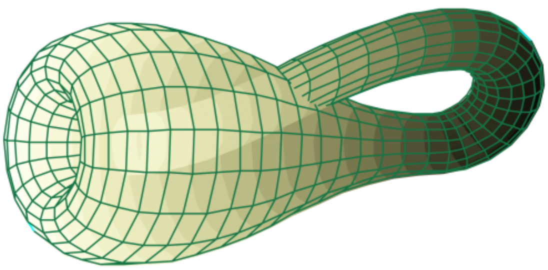
$\mathbb{Z}/2\mathbb{Z}$

$$\begin{tikzcd}
0 & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d] & 0 \\
& \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} &
\end{tikzcd}$$

$$(\underbrace{x_{n-} : x_{n-1} : \dots}_{k \to \mathbb{R}P^k} : \underline{x_0})$$



$H_*(\;\cdot\;) \cong ??$

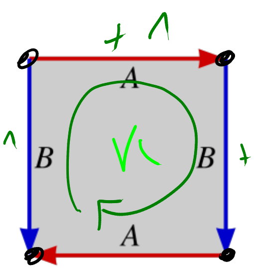


$H_*(\;\cdot\;, \underline{\mathbb{Z}/2\mathbb{Z}}) \cong ??$

??

$\langle A, B \mid ABAB^{-1} = 1 \rangle$

$[B] \rightsquigarrow$ trivial

$K =$ 

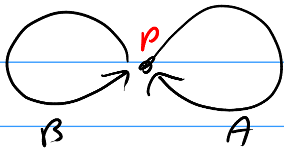
$\overset{?}{\leftrightsquigarrow} H_*(K, \mathbb{Z}/2\mathbb{Z}) = \underline{\mathbb{Z}/2\mathbb{Z}\{[P]\} \oplus t\mathbb{Z}/2\mathbb{Z}\{[A], [B]\} \oplus t^2\mathbb{Z}/2\mathbb{Z}\{[K]\}}$

$\hookrightarrow$ one 0-cell two 1-cells One 2-cell

$$\mathbb{Z} \xrightarrow{\binom{2}{0}} \mathbb{Z}^2 \xrightarrow{(0\ \ 0)} \mathbb{Z} \Rightarrow H_*(K) \cong \mathbb{Z} \oplus t(\underset{[B]}{\mathbb{Z}} \oplus \underset{[A]}{\mathbb{Z}/2\mathbb{Z}}) \qquad t^2\, 0$$

$$\boxed{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\binom{0}{0}} \boxed{\mathbb{Z}/2\mathbb{Z}^2} \xrightarrow{(0\ \ 0)} \boxed{\mathbb{Z}/2\mathbb{Z}} \Rightarrow H_*(K, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus t\mathbb{Z}/2\mathbb{Z}^{\oplus} \oplus t^2\mathbb{Z}/2\mathbb{Z}$$

$$\begin{array}{ccc} & A & B \\ p & (0 & 0) \end{array}$$



$$\begin{array}{cc} & D^2 \\ A & \left(\begin{matrix} 2 \\ 0 \end{matrix} \right. \\ B & \left. \begin{matrix} \\ \end{matrix} \right) \end{array}$$

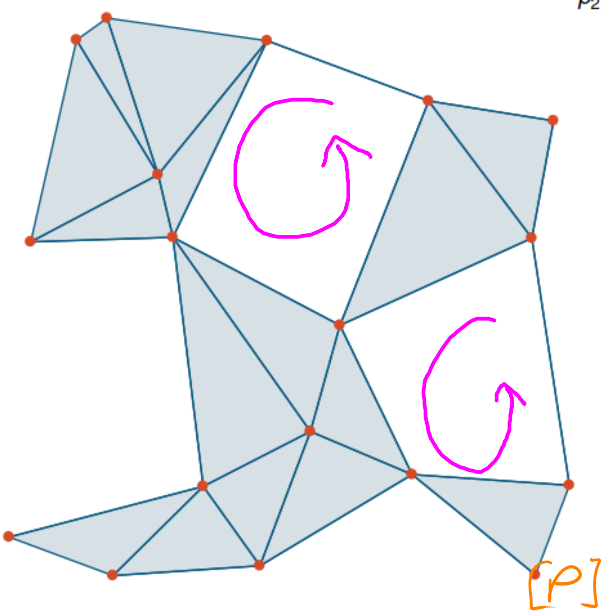
$$\mathbb{C}P^n \quad \to \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

$2n$

$$H_*(\mathbb{C}P^n) \simeq \underset{[\mathbb{C}P^0]}{\mathbb{Z}} \oplus t^2 \underset{[\mathbb{C}P^1]}{\mathbb{Z}} \oplus t^4 \underset{[\mathbb{C}P^2]}{\mathbb{Z}} \cdots \oplus t^{2n} \underset{[\mathbb{C}P^n]}{\mathbb{Z}}$$

$H_0 \cong \mathbb{Z}$ [p]
$H_1 \cong \mathbb{Z} \oplus \mathbb{Z}$
$H_2 \cong \{0\}$

$\beta_0 = 1$
$\beta_1 = 2$
$\beta_2 = 0$

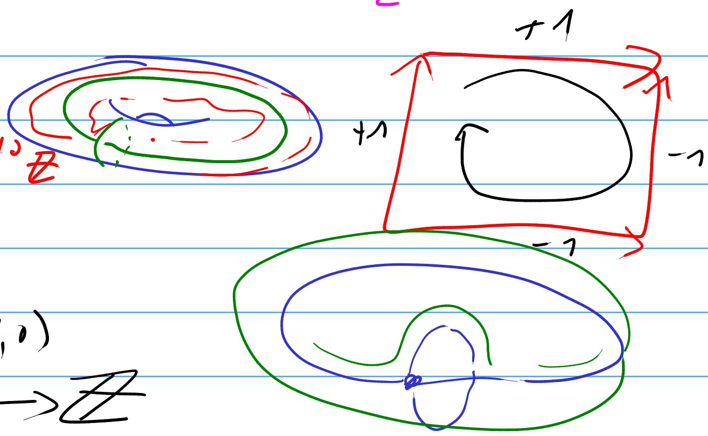


List of homologies:

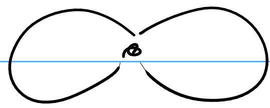
$$H_*(S^n) = \mathbb{Z} \oplus t^n \mathbb{Z}$$

$$\underbrace{\mathbb{Z}}_{t^n\mathbb{Z}} \longrightarrow 0 \ \cdots\cdots \ 0 \longrightarrow \underbrace{\mathbb{Z}}_{\mathbb{Z}}$$

$$H_*(T) \simeq \mathbb{Z} \oplus t\mathbb{Z}^2 \oplus t^2\mathbb{Z}$$



$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\begin{pmatrix} 0 \\ 0 \end{pmatrix}"] \arrow[d] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "{(0,0)}"] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$



$$H_*(T^n) \qquad \binom{n}{2} \qquad\qquad \binom{n}{1} \qquad\qquad \binom{n}{0}$$

$$\mathbb{Z} \cdots \mathbb{Z} \xrightarrow{0} \mathbb{Z} \cdots\cdots \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$$H_*(M_{g,0}) \simeq \underline{\mathbb{Z} \oplus t\,\mathbb{Z}^{2g} \oplus t^2\mathbb{Z}}$$

non-orientable

$$H_*(K) = \underline{\mathbb{Z} \oplus t(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})} \oplus \cancel{t^2} 0$$

$$H_*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}_2 \oplus t\,\mathbb{Z}_2 \oplus t^2\,\mathbb{Z}/2 \cdots$$

$$H_*(\mathbb{CP}^n) = \mathbb{Z} \oplus t^2\mathbb{Z} \oplus t^4\mathbb{Z} \oplus \cdots$$

orientable $\iff$ n is odd

Homology is a functor: its knows more than spaces, it also knows maps between spaces!

$$X \qquad \begin{array}{c} \text{An honest module} \\ \downarrow \\ \text{Graded dimensions} \\ \downarrow \\ t=-1 \end{array} \qquad \begin{array}{c} \text{homology } H_* \\ \downarrow \\ \text{Hilbert–Poincaré polynomial } P \\ \downarrow \\ \text{Euler } \chi \end{array}$$

$$f\colon X \to X \qquad \begin{array}{c} \text{An honest map} \\ \downarrow \\ \text{nothing} \\ \downarrow \\ \mathrm{tr} \end{array} \qquad \begin{array}{c} \text{homology } H_* \\ \downarrow \\ \text{really} \\ \downarrow \\ \text{Lefschetz } \Lambda \end{array} \quad \to f_*$$

Lefschetz number $\Lambda(f)$, $f\colon X \to X$ (everything appropriately finite) is

$$\Lambda(f) = \sum_i (-1)^i \mathrm{tr}(H_i(f)) = \sum_i (-1)^i \mathrm{tr}(C_i(f))$$

↑ Properties

Homotopy invariant

$$\chi(X) = \sum_i (-1)^i \dim H_i(X, \mathbb{Q})$$

$$\overset{\text{Prop.}}{=} \sum (-1)^i \dim \underbrace{C_i(X,\mathbb{Q})}_{\#\text{ cells}}$$

Theorem X finite cell complex, $f\colon X \to X$ has no fixed points $\Rightarrow$ $\Lambda(f) = 0$

If X is simply connected, closed manifold, then the converse is also true.

$S^2 \quad S^1$

$\begin{pmatrix}2\\0\end{pmatrix} 4 = \begin{pmatrix}8\\0\end{pmatrix}$ ✓

Assume $f\colon K \to K$ gives

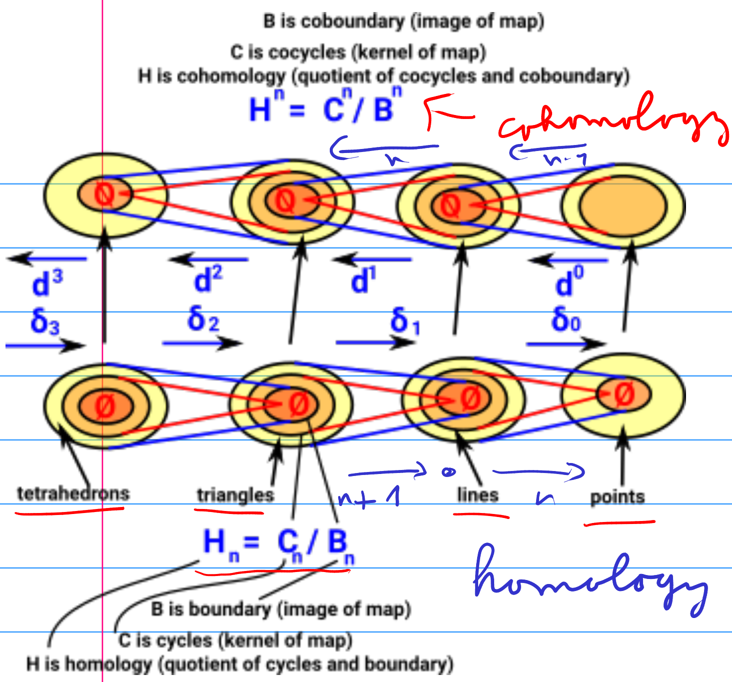
$$\begin{tikzcd} C_* \quad \mathbb{Z} \arrow[r, "\begin{pmatrix}2\\0\end{pmatrix}"] \arrow[d, "4 = f_*"] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix}0 & 0\end{pmatrix}"] \arrow[d, "\begin{pmatrix}4 & 1\\0 & -10\end{pmatrix} = f_*"] & \mathbb{Z} \arrow[d, "0 = f_*"] \\ C_* \quad \mathbb{Z} \arrow[r, "\begin{pmatrix}2\\0\end{pmatrix}"'] & \mathbb{Z}^2 \arrow[r, "\begin{pmatrix}0 & 0\end{pmatrix}"'] & \mathbb{Z} \end{tikzcd}$$

| f_5 | f_4 | f_3 | f_1 | f_0 |

$\begin{pmatrix}4 & 1\\0 & -10\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}8\\0\end{pmatrix}$ ✓

$(\Lambda(f) = \cancel{2}\ {-6}) \Rightarrow$ no fixed point

$\mathrm{tr}\, f_2 = 4 \quad \mathrm{tr}\, f_1 = \mp 6 \quad \mathrm{tr}\, f_0 = 0$



$H^* \rightsquigarrow$ ring $\quad \mathbb{Z}, \mathbb{Z}[x] \quad \mathbb{Z}[x]/x^2$

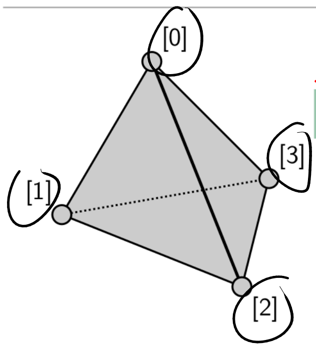
nice

$(\delta^*)^2 = 0$

$\delta_*^2 = 0$

$H_* \rightsquigarrow$ coring

Homology goes down



$[0,1,2,3] \mapsto [0,1,2] + [0,1,3] + [0,2,3] + [1,2,3]$

$[0,1,2] \mapsto [0,1] + [0,2] + [1,2]$

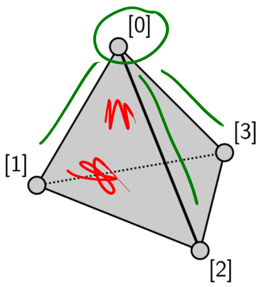
$[0,1] \mapsto [0] + [1]$

- tetrahedron $\xrightarrow{\text{homology}}$ sum of triangles $3 \to 2$
- triangle $\xrightarrow{\text{homology}}$ sum of lines $2 \to 1$
- line $\xrightarrow{\text{homology}}$ sum of points $1 \to 0$

(Here with $\mathbb{Z}/2\mathbb{Z}$ coefficients so no need to worry about orientations.)

$[0,1,2,3] \to [0,1,2] + [0,1,3] + [0,2,3] + [1,2,3]$

Cohomology goes up



$[0,1,2] \mapsto [0,1,2,3]$

$[0,1] \mapsto [0,1,2] + [0,1,3]$

$[0] \mapsto [0,1] + [0,2] + [0,3]$

- triangle $\xrightarrow{\text{cohomology}}$ sum of tetrahedrons $2 \to 3$
- line $\xrightarrow{\text{cohomology}}$ sum of triangles $1 \to 2$
- point $\xrightarrow{\text{cohomology}}$ sum of lines $0 \to 1$

(Here with $\mathbb{Z}/2\mathbb{Z}$ coefficients so no need to worry about orientations.)

Let X be any topological space $\to C_n(X)$

- The nth singular co chain group is
$$C^n = C^n(X) = \mathbb{Z}\{\text{singular } n\text{-cosimplices}\} = \hom(\mathbb{Z}\{\sigma_n \colon \Delta^n \to X\}, \mathbb{Z})$$
- The nth singular co chain map is
$$\partial^n \colon C^n \to C^{n-1}, \quad \partial^n = (\partial_n)^*$$
- The ith singular co homology is
$$H^n = H^n(X) = \ker(\partial^n)/\operatorname{im}(\partial^{n-1})$$ Homology has $\operatorname{im}(\partial_{n+1})$
- Singular cohomology is a homotopy/homeomorphism invariant

$C^n(X) = \hom(C_n, \mathbb{Z})$

↑

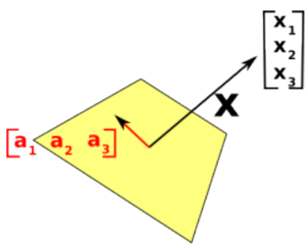
dual vector space

$* = T =$ transpose

Linear forms instead of vectors

- Note that
$$C^n = \hom(C_n, \mathbb{Z}), \quad \partial^n = (\partial_n)^*$$
This reverses all the arrows
- This is the same idea of defining dual vectors as linear forms

$(f \circ g)^* = g^* \circ f^*$



hom

$$\underbrace{(a_1, a_2, a_3)}_{\text{cohom}}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = a_1x_1 + a_2x_2 + a_3x_3 \in \mathbb{Z}$$

Transpose vectors

Simplicial and cellular cohomology also exists

Singular cohomology=simplicial cohomology=cellular cohomology for any reasonable X

cellcomplex

Note that we take $\hom = ()^*$ on the chain complex not the homology

$$0 \to \underset{2}{\mathbb{Z}} \xrightarrow[\delta_2]{\begin{pmatrix}2\\0\end{pmatrix}} \underset{1}{\mathbb{Z}^2} \xrightarrow[\delta_1]{\begin{pmatrix}0 & 0\end{pmatrix}} \underset{0}{\mathbb{Z}} \to 0 \Rightarrow H_*(K) \cong \mathbb{Z} \oplus t(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$$

$\#$

$$0 \leftarrow \underset{2}{\mathbb{Z}} \xleftarrow[(\delta_2)^*]{\begin{pmatrix}2 & 0\end{pmatrix}} \underset{1}{\mathbb{Z}^2} \xleftarrow[(\delta_1)^*]{\begin{pmatrix}0\\0\end{pmatrix}} \underset{0}{\mathbb{Z}} \leftarrow 0 \Rightarrow H^*(K) \cong \mathbb{Z} \oplus t\mathbb{Z} \oplus t^2\mathbb{Z}/2\mathbb{Z}$$

$$\mathbb{Q} \xrightarrow{\begin{pmatrix}2\\0\end{pmatrix}} \mathbb{Q}^2 \xrightarrow{\begin{pmatrix}0 & 0\end{pmatrix}} \mathbb{Q} \Rightarrow H_*(K) \cong \mathbb{Q} \oplus t\mathbb{Q}$$

K, Q

$$\mathbb{Q} \xleftarrow{\begin{pmatrix}2 & 0\end{pmatrix}} \mathbb{Q}^2 \xleftarrow{\begin{pmatrix}0\\0\end{pmatrix}} \mathbb{Q} \Rightarrow H^*(K) \cong \mathbb{Q} \oplus t\mathbb{Q}$$

K, Q

The difference between "first dual then homology" "first homology then dual" is measured by an exact sequence: The universal coefficient theorem (UCT) for cohomology for all X and PID R:

$$0 \to \mathrm{Ext}\big(H_{k-1}(X), R\big) \to H^k(X, R) \to \hom\big(H_k(X), R\big) \to 0$$

is a split (non-naturally) short exact sequence

$\mathbb{Q}$ ~ vanish

$(H_k)^*$

- Thus, in general
$$H^k(X) \cong \hom\big(H_k(X), \mathbb{Z}\big) \oplus \mathrm{Ext}\big(H_{k-1}(X), \mathbb{Z}\big)$$
- Ext vanishes over $\mathbb{Q}$ and $\hom\big(H_k(X), \mathbb{Q}\big) \cong H_k(X, \mathbb{Q})$ if finite, which implies
$$H_k(X, \mathbb{Q}) \cong H^k(X, \mathbb{Q})$$

coring

ring

A cohomology theory H^* satisfying the dimension axiom is a contravariant functor $H^* \colon \mathrm{Top}^2 \to \mathbb{Z}\mathrm{mod}$ from pairs of topological spaces to $\mathbb{Z}$-modules together with nat. trafos $\partial = \partial^n(X, A) \colon H^n(A) = H^n(A, \emptyset) \to H^{n+1}(X, A)$ satisfying:

- Homotopic maps induce the same map in cohomology Homotopy invariance
- If (X, A) is a pair and $U \subset A$ such that its closure is contained in the interior of A, then the inclusion
$$\iota \colon (X \setminus U, A \setminus U) \to (X, A)$$
induces an isomorphism in cohomology Excision
- Each (X, A) induces a long exact sequence
$$\ldots \to H^{n+1}(A) \xrightarrow{\partial} H^n(X, A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \to \cdots$$
via the inclusions $i \colon A \hookrightarrow X$ and $j \colon X \hookrightarrow (X, A)$ Exactness
- Direct products $\prod_i H^*(X_i)$ correspond to disjoint unions $\coprod_i X_i$: they are isomorphic by the inclusions $(\iota_i)^*$ $\prod \rightsquigarrow \coprod$
- $H^n(\text{point}) = 0$ for all $n > 0$, and $H^0(\text{point}) = \mathbb{Z}$ Dimension axiom

$\rightsquigarrow$ H^* is uniquely det by these properties

cell complexes

$f \colon X \to Y$

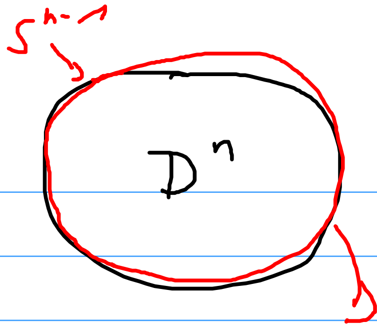
$H^* \rightsquigarrow f^* = H^*(f) \colon H^*(Y) \to H^*(X)$

$H^*(fg) = H^*(g)\, H^*(f)$

1

$H_k = \mathbb{Z}$-mod

$\hom(H_{k_1}, \mathbb{Z}) = (H_{k_2})^*$



$$D^n / S^{n-1} \cong S^{n-1}$$