Categorification and applications in topology and representation theory

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What is categorification?

Forced to reduce this presentation to one sentence, the author would choose:

Interesting integers are shadows of richer structures in categories.

The basic idea can be seen as follows. Take a "set-based" structure S and try to find a "category-based" structure $\mathcal{C}$ such that S is just a shadow of $\mathcal{C}$.

Categorification, which can be seen as "remembering" or "inventing" information, comes with an "inverse" process called decategorification, which is more like "forgetting" or "identifying".

Note that decategorification should be easy.

Exempli gratia

Examples of the pair categorification/decategorification are:

The integers $\mathbb{Z}$	$\underset{\text{decat}=\chi(\cdot)}{\overset{\text{categorify}}{\rightleftarrows}}$	complexes of VS
Polynomials in $\mathbb{Z}[q, q^{-1}]$	$\underset{\text{decat}=\chi_{\mathrm{gr}}(\cdot)}{\overset{\text{categorify}}{\rightleftarrows}}$	complexes of gr.VS
The integers $\mathbb{Z}$	$\underset{\text{decat}=K_0(\cdot)}{\overset{\text{categorify}}{\rightleftarrows}}$	$K-$ vector spaces
An $A-$ module	$\underset{\text{decat}=K_0^{\oplus}(\cdot)\otimes_{\mathbb{Z}}A}{\overset{\text{categorify}}{\rightleftarrows}}$	additive category

The first/second part is related to the first/last two examples.

Classical and virtual knots and links

Definition

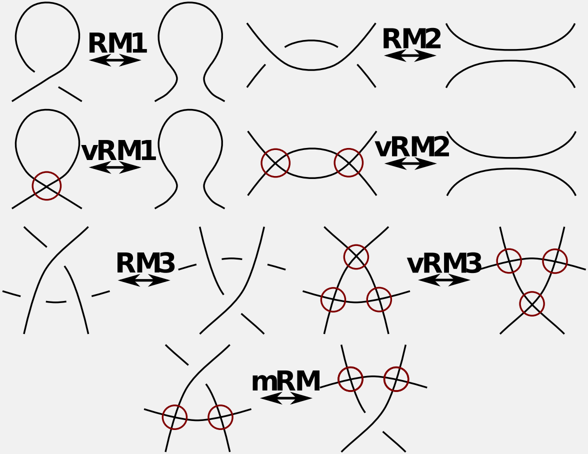
A virtual knot or link diagram L_D is a four-valent graph embedded in the plane. Moreover, every vertex is marked with an overcrossing , an undercrossing  or a virtual crossing .

An oriented virtual knot or link diagram is defined by orienting the projection, i.e. crossings should look like ,  and .

A virtual knot or link L is an equivalence class of virtual knot or link diagrams modulo the so-called generalised Reidemeister moves.

Classical and virtual knots and links

Generalised Reidemeister moves

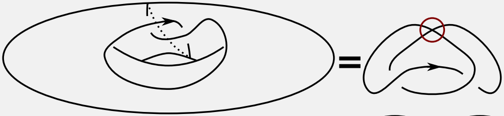


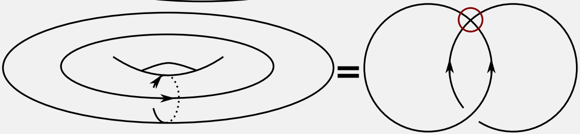
Virtual links and topology

Theorem(Kauffman, Kuperberg)

Virtual links are a combinatorial description of copies of S^1 embedded in a thickened surface Σ_g of genus g. Such links are equivalent iff their projections to Σ_g are stable equivalent, i.e. up to homeomorphisms of surfaces, adding/removing "unimportant" handles, classical Reidemeister moves and isotopies.

Example(Virtual trefoil and virtual Hopf link)





The famous (virtual) Jones polynomial

Let L_D be an oriented link diagram. The bracket polynomial $\langle L_D \rangle \in \mathbb{Z}[q, q^{-1}]$ can be recursively computed by the rules:

- $\langle \emptyset \rangle = 1$ (normalisation).
- $\langle \times \rangle = \langle)\,(\rangle - q\langle \asymp \rangle$ (recursion step 1).
- $\langle \text{Unknot} \amalg L_D \rangle = (q + q^{-1})\langle L_D \rangle$ (recursion step 2).

The Kauffman polynomial is $K(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$, with $n_+ =$number $\nearrow\!\!\!\!\nwarrow$ and $n_- =$number of $\nwarrow\!\!\!\!\nearrow$.

Theorem(Kauffman)

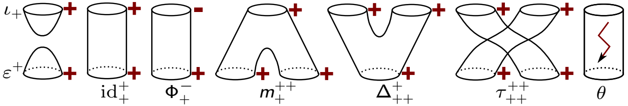
The Kauffman polynomial $K(L)$ is an invariant of virtual links and $K(L) = \hat{J}(K)$, where $\hat{J}(K)$ denotes the unnormalised Jones polynomial.

Let us categorify this!

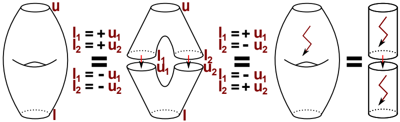
A cobordism approach

The pre-additive, monoidal, graded category $\mathbf{uCob}^2{}_R(\emptyset)$ of possible unorientable, decorated cobordisms has:

- Objects are resolutions of virtual link diagrams, i.e. virtual link diagrams without classical crossings.
- Morphisms are decorated cobordisms immersed into $\mathbb{R}^2 \times [-1,1]$ generated by (last one is a two times punctured $\mathbb{RP}^2$)



- Some relations like (last two are two times punctured Klein bottles)



- The monoidal structure is given by the disjoint union and the grading by the Euler characteristic.

How to form a chain complex

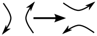
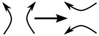
Define $\mathbf{Mat}(\mathbf{uCob}^2{}_R(\emptyset))$ to be the category of matrices over $\mathbf{uCob}^2{}_R(\emptyset)$, i.e. objects are formal direct sums of the objects of $\mathbf{uCob}^2{}_R(\emptyset)$ and morphisms are matrices whose entries are morphisms from $\mathbf{uCob}^2{}_R(\emptyset)$.

Define $\mathbf{uKob}_b(\emptyset)_R$ to be the category of chain complexes over $\mathbf{Mat}(\mathbf{uCob}^2{}_R(\emptyset))$. The category is pre-additive. Hence, the notion $d^2 = 0$ makes sense.

As a reminder, to every virtual link diagram L_D we want to assign an object in $\mathbf{uKob}_b(\emptyset)_R$ that is an invariant of virtual links. By our construction, this invariant will decategorify to the virtual Jones polynomial.

How to form a chain complex

For a virtual link diagram L_D with $n = n_+ + n_-$ crossings the topological complex $[\![L_D]\!]$ should be:

- For $i = 0, \ldots, n$ the $i - n_-$ chain module is the formal direct sum of all resolutions of length i.
- Between resolutions of length i and $i + 1$ the morphisms should be saddles between the resolutions.
- The decorations for the saddles can be read of by choosing an orientation for the resolutions. Locally they look like , which is called standard. Now compose with Φ iff the orientations differ or iff both are non-alternating  we use θ.
- Extra formal signs - placement is rather technical and skipped today.

Note that it not obvious why this definition gives a well-defined chain complex independent of all choices involved.

Some results

Theorem(s)(T)

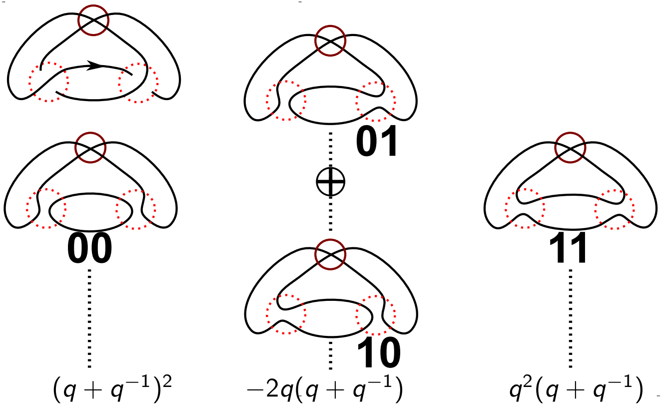
The topological complexes $[\![\cdot]\!]$ of two equivalent virtual link diagrams are the same in $\mathbf{uKob}_b(\emptyset)_R^{hl}$, i.e. the complex is an invariant up to chain homotopy and so-called local relations. Moreover, it is a well-defined chain complex independent of all choices involved and can be extended to virtual tangles.

Let $\mathcal{F}$ denote a uTQFT, i.e. a suitable functor $\mathcal{F}\colon \mathbf{uCob}^2{}_R(\emptyset) \to \text{R-}\mathbf{Mod}$.

Theorem(s)(T)

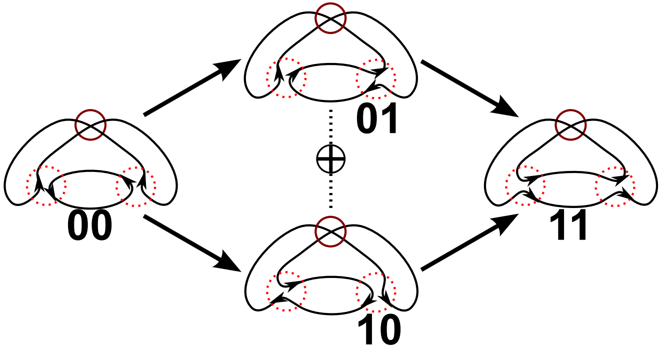
Let $\mathcal{F}$ be an uTQFT that satisfies the local relations. Then the homology groups of the algebraic complex $\mathcal{F}([\![\cdot]\!])$ are virtual link invariants. Moreover, the category of uTQFT is equivalent to the category of skew-extended Frobenius algebras.

Exempli gratia



Let us show how the calculation works. We consider the virtual trefoil and suppress grading shifts and sign placement. First let us add some orientations.

Exempli gratia



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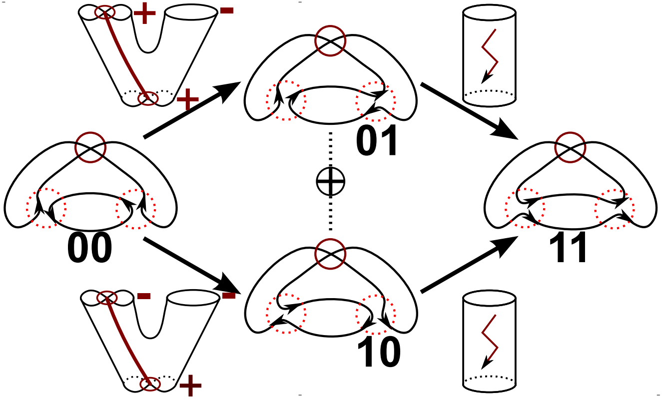

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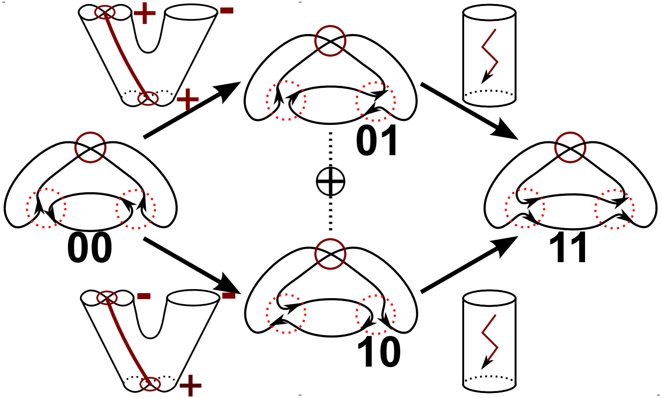
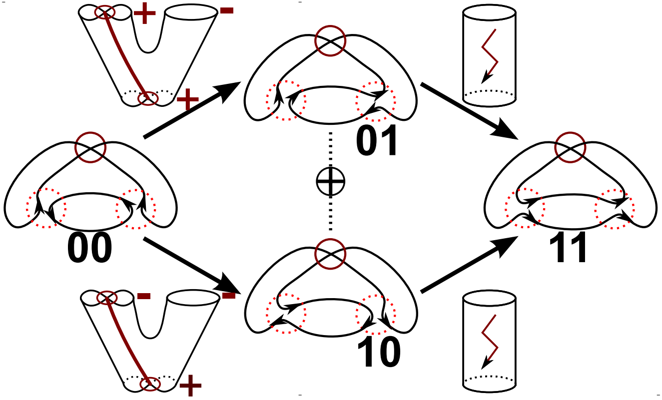


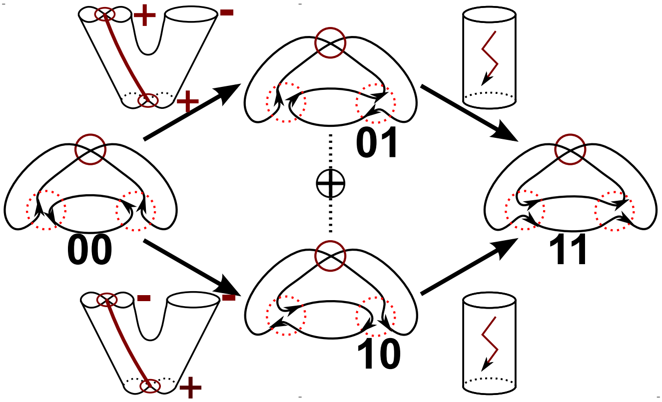
Note that this is the topological complex.

Exempli gratia



Now we have to translate (using one particular uTQFT) the objects to graded $\mathbb{Q}$-vector spaces and the cobordisms to $\mathbb{Q}$-linear maps between them. Then the objects are $A \otimes A$, $A \oplus A$ and A with $A = \mathbb{Q}[X]/X^2$.

Exempli gratia

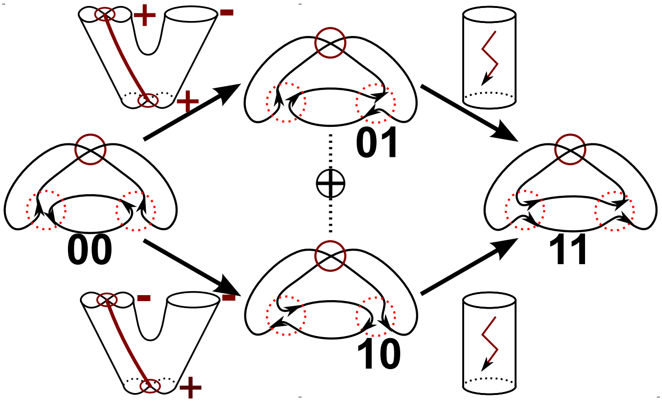


The two right maps are 0 and the two multiplications are given by

$$1 \otimes 1 \to 1, X \otimes 1 \to \pm X, 1 \otimes X \to -X \text{ and } X \otimes X \to 0$$

for the upper and lower. Note that they are not the same.

Exempli gratia

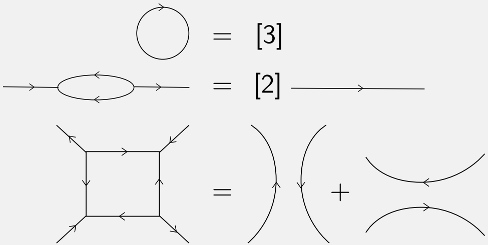


The homology can be computed now and it turns out to be (up to shifts) $q^{-2}t^{0} + q^{2}t^{-1} + qt^{-2} + q^{3}t^{-2}$. Setting $t = -1$ gives the virtual Jones polynomial $(q^{-1} - q + q^{2})(q + q^{-1})$.

Kuperberg's $\mathfrak{sl}_3$ webs

Definition(Kuperberg)

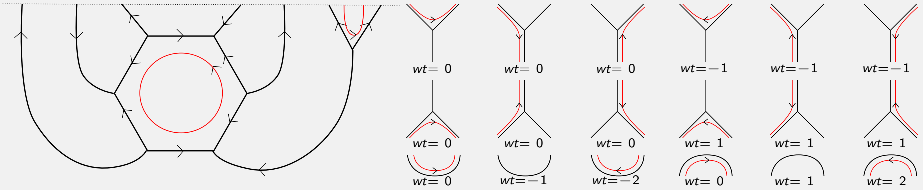
The $\mathbb{C}(q)$-web space W_S for a given sign string $S = (\pm, \dots, \pm)$ is generated by $\{w \mid \partial w = S\}$, where w is a web, i.e. an oriented, trivalent graph such that any vertex is either a sink or a source, with boundary S subject to the relations



Here $[a] = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \cdots + q^{-(a-1)}$ is the quantum integer.

Kuperberg's $\mathfrak{sl}_3$ webs

Example



Webs can be coloured with flow lines. At the boundary, the flow lines can be represented by a state string J. By convention, at the i-th boundary edge, we set $j_i = \pm 1$ if the flow line is oriented upward/downward and $j_i = 0$, if there is no flow line. So $J = (0, 0, 0, 0, 0, -1, 1)$ in the example.

Given a web with a flow w_f, attribute a weight to each trivalent vertex and each arc in w_f and take the sum. The weight of the example is -3.

Representation theory of $\mathbf{U}_q(\mathfrak{sl}_3)$

A sign string $S = (s_1, \dots, s_n)$ corresponds to tensors

$$V_S = V_{s_1} \otimes \cdots \otimes V_{s_n},$$

where V_+ is the fundamental representation and V_- is its dual, and webs correspond to intertwiners.

Theorem(Kuperberg)

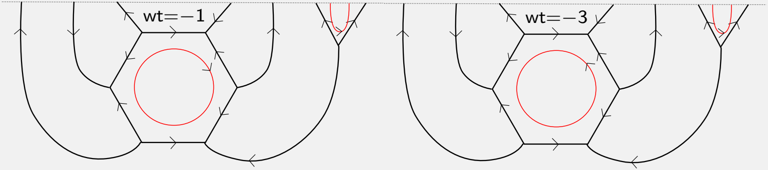
$$W_S \cong \hom_{\mathbf{U}_q(\mathfrak{sl}_3)}(\mathbb{C}(q), V_S) \cong \mathrm{Inv}_{\mathbf{U}_q(\mathfrak{sl}_3)}(V_S)$$

The set of non-elliptic webs, i.e. without circles, digons or squares, of W_S, denoted B_S, is called web basis of $\mathrm{Inv}_{\mathbf{U}_q(\mathfrak{sl}_3)}(V_S)$. In fact, the so-called spider category of all webs modulo the Kuperberg relations is equivalent to the representation category of $\mathbf{U}_q(\mathfrak{sl}_3)$.

Theorem(Khovanov, Kuperberg)

Pairs of sign S and a state strings J correspond to the coefficients of the web basis relative to tensors of the standard basis $\{e^{\pm}_{-1}, e^{\pm}_{0}, e^{\pm}_{+1}\}$ of $V_{\pm}$.

Example



$$w_S = \cdots - (q^{-1} + q^{-3})(e_0^+ \otimes e_0^- \otimes e_0^+ \otimes e_0^- \otimes e_0^+ \otimes e_{-1}^+ \otimes e_{+1}^+) \pm \cdots .$$

Let us categorify this!

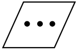
$\mathfrak{sl}_3$-foams

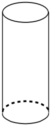
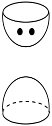
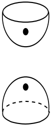
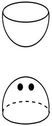
A pre-foam is a cobordism with singular arcs between two webs. Composition consists of placing one pre-foam on top of the other. The following are called the zip and the unzip respectively.

They have dots that can move freely about the facet on which they belong, but we do not allow dot to cross singular arcs.

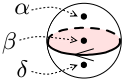
A foam is a formal $\mathbb{C}$-linear combination of isotopy classes of pre-foams modulo the following relations.

The foam relations $\ell = (3D, NC, S, \Theta)$

 $= 0$ (3D)

 $= -$  $-$  $-$  (NC)

 $=$  $= 0,$  $= -1$ (S)



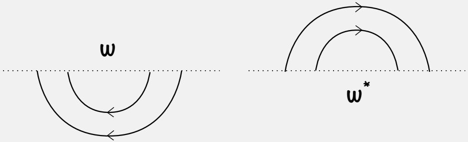
$$= \begin{cases} 1, & (\alpha,\beta,\delta) = (1,2,0) \text{ or a cyclic permutation,} \\ -1, & (\alpha,\beta,\delta) = (2,1,0) \text{ or a cyclic permutation,} \\ 0, & \text{else.} \end{cases} \qquad (\Theta)$$

Adding a closure relation to ℓ suffice to evaluate foams without boundary!

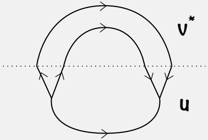
Involution on webs and closed webs

Definition

There is an involution * on the webs.



A closed web is defined by closing of two webs.



A closed foam is a foam from $\emptyset$ to a closed web.

The $\mathfrak{sl}_3$-foam category

Foam$_3$ is the category of foams, i.e. objects are webs w and morphisms are foams F between webs. The category is graded by the $q-$degree

$$\deg_q(F) = \chi(\partial F) - 2\chi(F) + 2d + b,$$

where d is the number of dots and b is the number of vertical boundary components. The foam homology of a closed web w is defined by

$$\mathcal{F}(w) = \mathbf{Foam}_3(\emptyset, w).$$

$\mathcal{F}(w)$ is a graded, complex vector space, whose q-dimension can be computed by the Kuperberg bracket.

The $\mathfrak{sl}_3$ web algebra

Definition(MPT)

Let $S = (s_1, \ldots, s_n)$. The $\mathfrak{sl}_3$ web algebra K_S is defined by

$$K_S = \bigoplus_{u,v \in B_S} {}_uK_v,$$

with

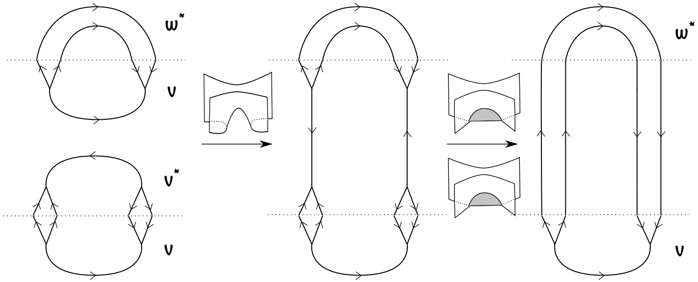
$${}_uK_v := \mathcal{F}(u^*v)\{n\}, \text{ i.e. all foams: } \emptyset \to u^*v.$$

Multiplication is defined as follows.

$${}_uK_{v_1} \otimes {}_{v_2}K_w \to {}_uK_w$$

is zero, if $v_1 \neq v_2$. If $v_1 = v_2$, use the multiplication foam m_v, e.g.

The $\mathfrak{sl}_3$ web algebra

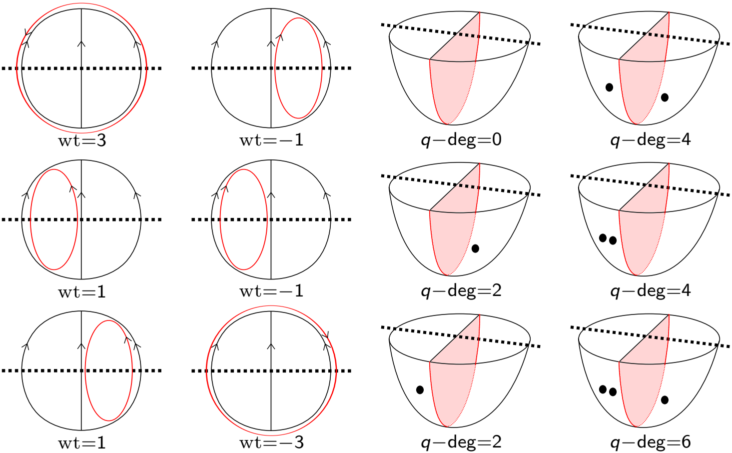


Theorem(s)(MPT)

The multiplication is associative and unital. The multiplication foam m_v only depends on the isotopy type of v and has q-degree n. Hence, K_S is a finite dimensional, unital and graded algebra. Moreover, it is a graded Frobenius algebra of Gorenstein parameter $2n$.

Exempli gratia

Every web has a homogeneous basis parametrised by flow lines.



That these foams are really a basis follows from a theorem of us. Note that the Kuperberg bracket gives $[2][3] = q^{-3} + 2q^{-1} + 2q + q^{3}$.

En(c)hanced sign strings

Definiton

An enhanced sign sequence is a sequence $S = (s_1, \dots, s_n)$ with $s_i \in \{\circ, -, +, \times\}$, for all $i = 1, \dots, n$. The corresponding weight $\mu = \mu_S \in \Lambda(n, d)$ is given by the rules

$$\mu_i = \begin{cases} 0, & \text{if } s_i = \circ, \\ 1, & \text{if } s_i = 1, \\ 2, & \text{if } s_i = -1, \\ 3, & \text{if } s_i = \times. \end{cases}$$

Let $\Lambda(n, d)_3 \subset \Lambda(n, d)$ be the subset of weights with entries between 0 and 3. Given S, we define $\widehat{S}$ by deleting the entries equal to $\circ$ or $\times$.

En(c)hanced sign strings

Moreover, for $n = d = 3^k$ we define

$$W_S = W_{\widehat{S}} \text{ and } B_S = B_{\widehat{S}} \text{ and } W_{(3^k)} = \bigoplus_{\mu_s \in \Lambda(n,n)_3} W_S$$

on the level of webs and on the level of foams, we define

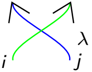
$$K_S = K_{\widehat{S}} \text{ and } \mathcal{W}_{(3^k)} = \bigoplus_{\mu_s \in \Lambda(n,n)_3} K_S - \mathrm{p}\mathbf{Mod}_{gr}.$$

With this constructions we obtain our categorification result.

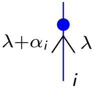
Theorem(MPT)

$$K_0^{\oplus}(\mathcal{W}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong W_{(3^k)}.$$

Khovanov and Lauda's diagrammatic categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, denoted $\mathcal{U}(\mathfrak{sl}_n)$, is also related to our framework! Roughly, it consist of string diagrams of the form



$: \mathcal{E}_i\mathcal{E}_j\mathbf{1}_\lambda \Rightarrow \mathcal{E}_j\mathcal{E}_i\mathbf{1}_\lambda\{(\alpha_i,\alpha_j)\},$



$: \mathcal{E}_i\mathbf{1}_\lambda \Rightarrow \mathcal{E}_i\mathbf{1}_\lambda\{(\alpha_i,\alpha_i)\}$

with a weight $\lambda \in \mathbb{Z}^{n-1}$ and suitable shifts and relations like

Connection to $\mathbf{U}_q(\mathfrak{sl}_n)$

Let $\lambda \in \Lambda(n,n)^+$ be a dominant weight. Define the cyclotomic KL-R algebra R_λ to be the subquotient of $\mathcal{U}(\mathfrak{sl}_n)$ defined by the subalgebra of only downward pointing arrows modulo the so-called cyclotomic relations and set $\mathcal{V}_\lambda = R_\lambda - \mathrm{p}\mathbf{Mod}_{gr}$.

Theorem(s)(MPT)

There exists an equivalence of categorical $\mathcal{U}(\mathfrak{sl}_n)$-representations

$$\Phi\colon \mathcal{V}_{(3^k)} \to \mathcal{W}_{(3^k)}.$$

The two algebras $R_{3\ell}$ and $K_{3\ell}$ are Morita equivalent. Moreover, the set

$$\{[Q_u] \mid Q_u \text{ graded, indecomposable, projective } K_S - \text{module}, u \in B_S\}$$

is the dual canonical basis for $\mathrm{Inv}_{\mathbf{U}_q(\mathfrak{sl}_3)}(V_S) \cong K_0^{\oplus}(K_S) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q)$.

There is still much to do...

Thanks for your attention!