The 2-representation theory of di- and trihedral Soergel bimodules

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Section outline

Background

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• The positive integral properties follow from **categorification**.

• Furthermore, these algebras have representations with positive integrality properties. The most interesting ones are decategorifications of **2-representations**.

• The "simplest" 2-representations are the ones that are **simple transitive** (Mazorchuk-Miemietz 2014).

Plan

- Today we concentrate on

$$[U_\eta(\mathfrak{sl}_r)\mathrm{-mod}_{\mathrm{ss}}]_{\mathbb{C}} \quad \text{and} \quad \mathrm{H}(I_r(e+r)),$$

with $\eta^{2(e+r)} = 1$, for $r = 2, 3$.

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- **Disclaimer**: despite the title of my talk, I will mostly stick to the decategorified story.

Section outline

1. Introduction

2. $m = 2$
   - Quantum $\mathfrak{sl}_2$
   - The dihedral Hecke algebra

3. $m = 3$
   - Quantum $\mathfrak{sl}_3$
   - The trihedral Hecke algebra

Chebyshev polynomials

Definition (Chebyshev polynomials)

The polynomials $U_n(x)$, $n \in \mathbb{N}$, can be defined recursively:

$$U_0(x) = 1,\ U_1(x) = x,\ xU_n(x) = U_{n+1}(x) + U_{n-1}(x).$$

E.g.

$$\begin{aligned} U_2(x) &= x^2 - 1 \\ U_3(x) &= x^3 - 2x \\ U_4(x) &= x^4 - 3x^2 + 1 \end{aligned}$$

$$xU_3(x) = U_4(x) + U_2(x)$$

Chebyshev polynomials

We also have

$$U_n(2\cos(z)) = \frac{\sin((n+1)z)}{\sin(z)}$$

Lemma

$$U_n(x) = 0 \quad \Leftrightarrow \quad x = 2\cos\left(\frac{k\pi}{n+1}\right) \quad k = 1, \ldots, n.$$

Note that the zeros of the U_n all belong to $]-2, 2[$.

Relation with quantum $\mathfrak{sl}_2$: generic case

Let $\mathrm{q} \in \mathbb{C}$ be generic. For $n \in \mathbb{N}$, let V_n be the $(n+1)$-dimensional simple of $U_{\mathrm{q}}(\mathfrak{sl}_2)$.

Theorem

For any $n \in \mathbb{Z}_{\geq 1}$, we have

$$[V_1][V_n] = [V_{n+1}] + [V_{n-1}]$$

in the Grothendieck group $[U_{\mathrm{q}}(\mathfrak{sl}_2) - \mathrm{mod}]_{\mathbb{C}}$.

Thus the $[V_n]$ satisfy the same recursion relation as the U_n.

Relation with quantum $\mathfrak{sl}_2$: generic case

Theorem

There exists an isomorphism of algebras:

$$
\begin{aligned}
[U_{\mathrm{q}}(\mathfrak{sl}_2) - \mathrm{mod}]_{\mathbb{C}} &\quad\cong\quad \mathbb{C}[x] \\
[V_n] = \sum_{k=0}^{n} d_n^k \left[V_1^{\otimes k}\right] &\quad\mapsto\quad U_n(x) = \sum_{k=0}^{n} d_n^k x^k
\end{aligned}
$$

The integers d_n^k can be computed recursively. Note that they can be positive or negative.

Relation with quantum $\mathfrak{sl}_2$: root of unity case

Theorem

Suppose $\eta^{2(e+2)} = 1$. Then there exists an isomorphism of algebras

$$\begin{array}{rcll} [U_\eta(\mathfrak{sl}_2) - \mathrm{mod}_{\mathrm{ss}}]_{\mathbb{C}} & \cong & \mathbb{C}[x]/\left(U_{e+1}(x)\right) & (\textit{Verlinde algebra}) \\ [V_n] & \mapsto & U_n(x) & (0 \leq n \leq e). \end{array}$$

The dihedral group

Definition

The Coxeter group W_n of type $I_2(n)$ (dihedral group of order $2n$) is generated by s and t such that

$$s^2 = t^2 = (st)^n = 1.$$

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The last relation can be rewritten:

$$(st)^n = 1 \quad \Leftrightarrow \quad n_s = n_t \ \ (=: w_0),$$

e.g. for $n = 4$

$$stststst = e \quad \Leftrightarrow \quad tsts = stst \quad \Leftrightarrow \quad 4_s = 4_t.$$

Hecke algebra of dihedral type

Definition (Hecke algebra)

$\mathrm{H}(I_2(n))$ is the $\mathbb{C}(v)$-algebra generated T_s and T_t s.t.

$$T_s^2 = (\mathrm{v}^{-2} - 1) \cdot T_s + \mathrm{v}^{-2} \tag{1}$$

$$T_t^2 = (\mathrm{v}^{-2} - 1) \cdot T_t + \mathrm{v}^{-2} \tag{2}$$

$$T_{n_s} = T_{n_t}. \tag{3}$$

Here $T_{n_s} = \cdots T_t T_s$ and $T_{n_t} = \cdots T_s T_t$ are products of n alternating generators.

Regular basis

Definition and Theorem

The set

$$\{T_w \mid w \in W_n\}$$

forms a basis, called the **regular basis**.

We see:

$$\mathrm{H}(I_2(n)) = \mathbb{C}[W_n] \quad \text{for } \mathrm{v} = 1.$$

Kazhdan-Lusztig basis

The **Kazhdan-Lusztig basis** $\{\theta_w \mid w \in \mathrm{W}_n\}$.

Example

$\theta_s = \mathrm{v}(T_s + 1) \quad \theta_t = \mathrm{v}(T_t + 1) \qquad \theta_s\theta_t\theta_s = \theta_{sts} + \theta_s.$

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Positive integrality property:

- On the KL basis, all multiplication constants belong to $\mathbb{N}[\mathrm{v}, \mathrm{v}^{-1}]$.

Bott-Samelson basis

The **Bott-Samelson basis** $\{\theta_{\overline{w}} \mid w \in \mathrm{W}_n\}$.

Example

$$\begin{aligned} \theta_{\overline{sts}} &= \theta_s\theta_t\theta_s = \theta_{sts} + \theta_s, \\ \theta_{\overline{tst}} &= \theta_t\theta_s\theta_t = \theta_{sts} + \theta_t. \end{aligned}$$

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Facts:

- $\theta_{\overline{w_0}}$ depends on the choice $w_0 = n_s$ or $w_0 = n_t$.
- the change-of-basis between the KL-basis and the BS-basis:

$$\theta_{n_s} = \sum_{k=0}^{n} d_n^k \theta_{\overline{k_s}}, \qquad \theta_{n_t} = \sum_{k=0}^{n} d_n^k \theta_{\overline{k_t}}.$$

Hecke algebra of type $I_2(n)$

Proposition

$\mathrm{H}(I_2(n))$ is generated by θ_s and θ_t, subject to the relations

$$\theta_s\theta_s = [2]_{\mathrm{v}} \cdot \theta_s, \qquad \theta_t\theta_t = [2]_{\mathrm{v}} \cdot \theta_t$$

$$\sum_{k=0}^{n} d_n^k \cdot \theta_{\overline{k_s}} = \sum_{k=0}^{n} d_n^k \cdot \theta_{\overline{k_t}}.$$

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Note that $\theta_{w_0} = \sum_{k=0}^{n} d_n^k \cdot \theta_{\overline{k_s}} = \sum_{k=0}^{n} d_n^k \cdot \theta_{\overline{k_t}}$.

Definition (Small quotient)

The **small quotient** of $\mathrm{H}(I_2(n))$ is defined by

$$\overline{\mathrm{H}}(I_2(n)) := \mathrm{H}(I_2(n)) / (\theta_{w_0}).$$

Quantum Satake Correspondence ($\eta^{2(e+2)} = 1$)

- $\overline{\mathrm{H}}(I_2(e+2))$ is **almost** a bicolored $[U_\eta(\mathfrak{sl}_2)\mathrm{-mod}_{\mathrm{ss}}]_{\mathbb{C}}$:

$$[V_1]^k \leftrightsquigarrow \begin{cases} \theta_{\overline{k+1}_s} \\ \theta_{\overline{k+1}_t} \end{cases} \quad \textbf{(Bott-Samelson)}$$

$$[V_k] \leftrightsquigarrow \begin{cases} \theta_{k+1_s} \\ \theta_{k+1_t} \end{cases} \quad \textbf{(Kazhdan-Lusztig)}$$

$$[V_1]^2 = [V_2] + [V_0] \leftrightsquigarrow \begin{cases} \theta_s\theta_t\theta_s = \theta_{sts} + \theta_s \\ \theta_t\theta_s\theta_t = \theta_{tst} + \theta_t. \end{cases}$$

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- But **not quite**:

$$[V_0]^2 = [V_0] \not\leftrightsquigarrow \begin{cases} \theta_s^2 = [2]_{\mathrm{v}}\theta_s \\ \theta_t^2 = [2]_{\mathrm{v}}\theta_t \end{cases}$$

Irreducible representations of the Verlinde algebra

The characters of $\mathbb{C}[x]/\left(U_{e+1}(x)\right)$ are given by:

$$x \mapsto 2\cos\left(\frac{k\pi}{e+2}\right) \quad k = 1, \ldots, e+1.$$

Note that the corresponding irreducible representations are **not positive integral** in general.

Positive integral representations of the Verlinde algebra

i

Theorem (Coxeter 1951, Happel-Preiser-Ringel 1980, Etingof-Khovanov 1994)

Let $X \in \mathrm{Mat}(k, \mathbb{N})$. Then $x \mapsto X$ defines a positive integral representation of $\mathbb{C}[x]/(U_{e+1}(x))$ if(f)

$$X = 2I - A,$$

where A is the Cartan matrix of a finite type Dynkin or tadpole diagram with Coxeter number $h = e + 2$.

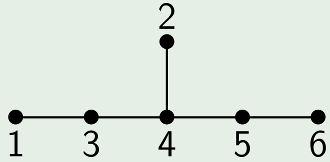
Remark:

- In type ADE, positivity can be proved by **categorification**.

ADE type representations

Let A be the Cartan matrix of a Dynkin diagram Γ of ADE type with $h = e + 2$. Then $2I - A$ is the **adjacency matrix** of Γ.

Example ($E_6, h = 12$)



$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Weak categorification

Let Γ be of type ADE with Coxeter number $h = e + 2$.
Define the commutative quiver algebra

$$Q_\Gamma := \bigoplus_{i \in V(\Gamma)} \mathbb{C}e_i \cong \mathbb{C}^{|V(\Gamma)|}$$

associated to the trivial quiver with vertex set $V(\Gamma)$ and no edges.

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associated to the trivial quiver with vertex set $V(\Gamma)$ and no edges. Then

$$B_\Gamma := \bigoplus_{i - j} P_i \otimes {}_jP$$

is a biprojective $Q-Q$-bimodule, where $P_i = Q_\Gamma e_i$ and ${}_iP = e_i Q_\Gamma$.

Weak categorification

Define the $\mathbb{C}$-linear functor $F_\Gamma \colon Q_\Gamma{-}\mathrm{fpmod} \ \to \ Q_\Gamma{-}\mathrm{fpmod}$:

$$M \mapsto B_\Gamma \otimes_{\mathbb{Q}_\Gamma} M.$$

Theorem

Let $\eta^{2(e+2)} = 1$. *Then*

$$[V_1] \mapsto F_\Gamma$$

gives a functorial representation ϕ_Γ *of* $[U_\eta(\mathfrak{sl}_2) - \mathrm{mod}_{\mathrm{ss}}]_{\mathbb{C}}$ *on* $Q_\Gamma - \mathrm{fpmod}$.

Kirillov-Ostrik 2001: ϕ_Γ can be extended to a 2-representation of $U_\eta(\mathfrak{sl}_2) - \mathrm{mod}_{\mathrm{ss}}$ on $Q_\Gamma - \mathrm{fpmod}$.

Complex simples of $\mathrm{H}(I_2(n))$

1-dimensional simples: for $\epsilon_i \in \{0, [2]_{\mathrm{v}}\}$ s.t. braid relation hold,

$$\theta_s \mapsto \epsilon_1 \quad \text{and} \quad \theta_t \mapsto \epsilon_2.$$

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2-dimensional simples: for $x \in]-2, 2[$ s.t. $U_{n-1}(x) = 0$,

$$\theta_s \mapsto \begin{pmatrix} [2]_{\mathrm{v}} & x \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad t \mapsto \begin{pmatrix} 0 & 0 \\ x & [2]_{\mathrm{v}} \end{pmatrix}.$$

Positive integral representations of $\overline{\mathrm{H}}_n$

Let $X = 2I - A \in \mathrm{Mat}(k, \mathbb{N})$, with A a Cartan matrix of type ADE and Coxeter number $h = e + 2$.

Theorem (Kildetoft-M-Mazorchuk-Zimmermann 2016)

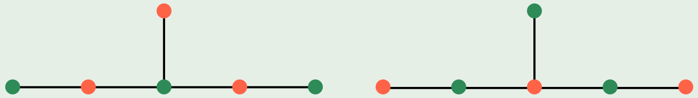
$$\theta_s \mapsto \begin{pmatrix} [2]_{\mathrm{v}} I & X \\ 0 & 0 \end{pmatrix} \qquad \theta_t \mapsto \begin{pmatrix} 0 & 0 \\ X & [2]_{\mathrm{v}} I \end{pmatrix}$$

defines a positive integral representation of $\overline{\mathrm{H}}(I_2(e+2))$.

Bipartite graphs and double quivers

Let Γ be a Dynkin diagram of type ADE with Coxeter number $h = e + 2$ and a bipartition.

Example ($E_6, h = 12$)



Define the **double quiver** DQ_Γ, e.g.

$$\Gamma = \underline{1} \longrightarrow\!\!\!\!\!\!\!\!- \overline{2} \text{——} \underline{3} \rightsquigarrow DQ_\Gamma = \underline{1} \rightleftarrows \overline{2} \rightleftarrows \underline{3}\ .$$

Zig-zag algebras

Definition (???, Huerfano-Khovanov 2000)

The **zig-zag algebra** Z_Γ is the quotient of the path algebra of DQ_Γ by the following relations:

$$\mathrm{i} \longrightarrow \mathrm{j} \longrightarrow \mathrm{i}' = 0, \qquad \mathrm{i} \longrightarrow \mathrm{j} \longrightarrow \mathrm{i} = \mathrm{i} \longrightarrow \mathrm{j}' \longrightarrow \mathrm{i} = \lambda \cdot \mathrm{i}|\mathrm{i},$$

$$\mathrm{j} \longrightarrow \mathrm{i} \longrightarrow \mathrm{j}' = 0, \qquad \mathrm{j} \longrightarrow \mathrm{i} \longrightarrow \mathrm{j} = \mathrm{j} \longrightarrow \mathrm{i}' \longrightarrow \mathrm{j} = \lambda \cdot \mathrm{j}|\mathrm{j}.$$

Facts:

- λ is a scalar which depends on η, where $\eta^{2(e+2)} = 1$.
- Z_Γ is graded by the path length.

Functorial representations

The biprojective bimodules

$$\bigoplus_{\underline{\mathrm{i}}\in\Gamma} P_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}P\{-1\} \qquad \bigoplus_{\bar{\mathrm{j}}\in\Gamma} P_{\bar{\mathrm{j}}} \otimes {}_{\bar{\mathrm{j}}}P\{-1\}$$

give rise to $\mathbb{C}$-linear endofunctors $\boldsymbol{\Theta}_s, \boldsymbol{\Theta}_t$ on $Z_\Gamma\mathrm{-fpmod_{gr}}$.

Theorem (KMMZ 2016, MT 2016)

The assignment

$$\theta_s \mapsto \boldsymbol{\Theta}_s \qquad \theta_t \mapsto \boldsymbol{\Theta}_t.$$

gives a functorial representation of $\overline{\mathrm{H}}(I_2(e+2))$ *on* $Z_\Gamma\mathrm{-fpmod_{gr}}$.

KMMZ 2016, MT 2016: this can be lifted to a 2-representation of (the small quotient of) dihedral Soergel bimodules.

Categorical intermezzo

- By the QSC, there is a precise correspondence between the simple transitive 2-representations of $U_\eta(\mathfrak{sl}_2) - \mathrm{mod}_{\mathrm{ss}}$ and those of the small quotient of the **maximally singular** dihedral Soergel bimodules.

Categorical intermezzo

- By the QSC, there is a precise correspondence between the simple transitive 2-representations of $U_\eta(\mathfrak{sl}_2) - \mathrm{mod}_{\mathrm{ss}}$ and those of the small quotient of the **maximally singular** dihedral Soergel bimodules.
- By **biinduction**, we get simple transitive 2-representations of the small quotient of **regular** dihedral Soergel bimodules.

Categorical intermezzo

- By the QSC, there is a precise correspondence between the simple transitive 2-representations of $U_\eta(\mathfrak{sl}_2) - \mathrm{mod}_{\mathrm{ss}}$ and those of the small quotient of the **maximally singular** dihedral Soergel bimodules.
- By **biinduction**, we get simple transitive 2-representations of the small quotient of **regular** dihedral Soergel bimodules.
- However, the simple transitive 2-representations of $U_\eta(\mathfrak{sl}_2) - \mathrm{mod}_{\mathrm{ss}}$ are **semisimple**, whereas those of the small quotient of regular dihedral Soergel bimodules are **not**!

Section outline

Chebyshev-like orthogonal polynomials

Definition (Chebyshev-like orthogonal polynomials)

The polynomials $U_{m,n}(x,y)$, $m,n \in \mathbb{N}$, are recursively defined by

$$U_{0,0}(x,y)=1,\ U_{1,0}(x,y)=x,\ U_{m,n}(x,y)=U_{n,m}(y,x),$$
$$xU_{m,n}(x,y)=U_{m+1,n}(x,y)+U_{m-1,n+1}(x,y)+U_{m,n-1}(x,y),$$
$$yU_{m,n}(x,y)=U_{m,n+1}(x,y)+U_{m+1,n-1}(x,y)+U_{m-1,n}(x,y).$$

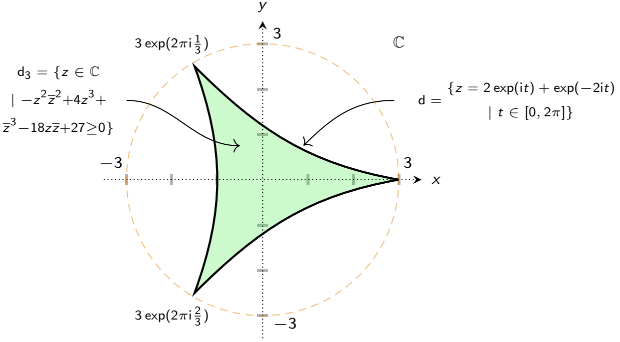
E.g.

$$U_{1,1}(x,y)=xy-1,\ U_{0,2}(x,y)=y^2-x$$
$$U_{2,1}(x,y)=x^2y-y^2-x$$

$$xU_{1,1}(x,y)=U_{2,1}(x,y)+U_{0,2}(x,y)+U_{1,0}(x,y)$$

The zeros of the $U_{m,n}$

The zeros of the $U_{m,n}$ are all of the form $(z, \overline{z}) \in \mathrm{d}_3^{\circ}$ (Koornwinder 1974, ...).



(4)

The disciod $\mathrm{d}_3 = \mathrm{d}_3(\mathfrak{sl}_3)$ bounded by Steiner's hypocycloid d

Note the $\mathbb{Z}/3\mathbb{Z}$-symmetry of d_3: $(z, \overline{z}) \mapsto (e^{2\pi i/3}z, e^{-2\pi i/3}\overline{z})$.

Relation with quantum $\mathfrak{sl}_3$: generic case

Let $\mathrm{q} \in \mathbb{C}$ be generic.

Theorem

There exists an isomorphism of algebras:

$$\begin{array}{rcl} [U_{\mathrm{q}}(\mathfrak{sl}_3) - \mathrm{mod}]_{\mathbb{C}} & \cong & \mathbb{C}[x, y] \\ [V_{m,n}] = \sum_{k,l=0}^{m,n} d_{m,n}^{k,l} \left[V_{1,0}^{\otimes k} \otimes V_{0,1}^{\otimes l} \right] & \mapsto & U_{m,n}(x, y) = \sum_{k,l=0}^{m,n} d_{m,n}^{k,l} x^k y^l \end{array}$$

for $m, n \in \mathbb{N}$.

The integers $d_{m,n}^{k,l}$ can be computed recursively. Note that they can be positive or negative.

Relation quantum $\mathfrak{sl}_3$: root of unity case

Theorem

Suppose $\eta^{2(e+3)} = 1$. Then there exists an isomorphism of algebras

$$\begin{aligned} [U_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}]_{\mathbb{C}} &\cong \mathbb{C}[x,y]/\left(U_{m,n}(x,y) \mid m+n = e+1\right) \\ [V_{m,n}] &\mapsto U_{m,n}(x,y) \quad (0 \leq m+n \leq e). \end{aligned}$$

The infinite-dimensional trihedral Hecke algebra

Definition (M-Mazorchuk-Miemietz-Tubbenhauer 2018)

Let $\mathrm{H}(I_3(\infty))$ be the associative, unital ($\mathbb{C}_{\mathrm{v}}$-)algebra generated by three elements θ_g, θ_o, θ_p, subject to the following relations.

$$\theta_g^2 = [3]_{\mathrm{v}}!\,\theta_g, \qquad \theta_o^2 = [3]_{\mathrm{v}}!\,\theta_o, \qquad \theta_p^2 = [3]_{\mathrm{v}}!\,\theta_p,$$

$$\theta_g\theta_o\theta_g = \theta_g\theta_p\theta_g, \qquad \theta_o\theta_g\theta_o = \theta_o\theta_p\theta_o, \qquad \theta_p\theta_g\theta_p = \theta_p\theta_o\theta_p.$$

Fact: $\mathrm{H}(I_3(\infty))$ is a subalgebra of $\mathrm{H}(\widehat{A}_2)$.

The trihedral Bott-Samelson basis

We fix a cyclic ordering:
$$\begin{tikzcd} p \arrow[dr, dotted] & & o \arrow[ll, dashed] \\ & g \arrow[ur] & \end{tikzcd}$$
. Then we have the *trihedral Bott-Samelson basis* $\{1\} \cup \left\{ \mathrm{h}_{\mathbf{u}}^{k,l} \mid \mathbf{u} \in \{g, o, p\},\ m, n \in \mathbb{N} \right\}$ of $\mathrm{H}(I_3(\infty))$.

Example

$$\underset{\leftrightsquigarrow\ x^2}{\mathrm{h}_g^{2,0} = \theta_p\theta_o\theta_g}, \quad \underset{\leftrightsquigarrow\ xy = yx}{\mathrm{h}_g^{1,1} = \theta_g\theta_p\theta_g = \theta_g\theta_o\theta_g}, \quad \underset{\leftrightsquigarrow\ y^2}{\mathrm{h}_g^{0,2} = \theta_o\theta_p\theta_g},$$

where we think of x and y as counter-clockwise and clockwise color rotation, resp.

The trihedral Kazhdan-Lusztig basis

For any $\mathbf{u} \in \{g, o, p\}$ and $m, n \in \mathbb{N}$, define

$$\mathrm{c}_{\mathbf{u}}^{m,n} := \sum_{k,l=0}^{m,n} [2]_{\mathrm{v}}^{-k-l} \, d_{m,n}^{k,l} \, \mathrm{h}_{\mathbf{u}}^{k,l}.$$

Proposition

The set

$$\{1\} \cup \{\mathrm{c}_{\mathbf{u}}^{m,n} \mid \mathbf{u} \in \{g, o, p\},\ m, n \in \mathbb{N}\}$$

forms a basis of $\mathrm{H}(I_3(\infty))$.

Fact: the embedding $\mathrm{H}(I_3(\infty)) \hookrightarrow \mathrm{H}(\hat{A}_2)$ preserves KL-elements.

The trihedral Hecke algebra of level e

Definition (M-Mazorchuk-Miemietz-Tubbenhauer 2018)

For fixed level e, let I_e be the two-sided ideal in $\mathrm{H}(I_3(\infty))$ generated by

$$\{c_{\mathbf{u}}^{m,n} \mid m+n = e+1,\ \mathbf{u} \in GOP\}.$$

We define the **trihedral Hecke algebra of level** e as

$$\mathrm{H}(I_3(e+3)) = \mathrm{H}(I_3(\infty))/\mathrm{I}_e.$$

Theorem

The algebra $\mathrm{H}(I_3(e+3))$ *is semisimple and*

$$\dim \mathrm{H}(I_3(e+3)) = 3\frac{(e+1)(e+2)}{2} + 1.$$

Complex simples of $\mathrm{H}(I_3(e+3))$

1-dimensional simples: for $\epsilon_{\mathbf{u}} \in \{0, [3]_{\mathrm{v}}!\}$ s.t. relations hold.

Complex simples of $\mathrm{H}(I_3(e+3))$

1-dimensional simples: for $\epsilon_{\mathbf{u}} \in \{0, [3]_{\mathrm{v}}!\}$ s.t. relations hold.

3-dimensional simples: for $0 \neq z \in \mathrm{d}_3^\circ$ s.t. $U_{m,n}(z, \overline{z}) = 0$ for all $m + n = e + 1$,

$$\theta_{g} \mapsto [2]_{\mathrm{v}} \begin{pmatrix} [3]_{\mathrm{v}} & \overline{z} & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \theta_{o} \mapsto [2]_{\mathrm{v}} \begin{pmatrix} 0 & 0 & 0 \\ z & [3]_{\mathrm{v}} & \overline{z} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\theta_{p} \mapsto [2]_{\mathrm{v}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \overline{z} & z & [3]_{\mathrm{v}} \end{pmatrix}.$$

Two 3-dimensional simples are isomorphic iff $z_1 = e^{\pm 2\pi i/3} z_2$.

Tricolored graphs

Let $\mathbf{\Gamma}$ be a tricolored graph and group its vertices according to color. Then the adjacency matrix $A(\mathbf{\Gamma})$ becomes of the form:

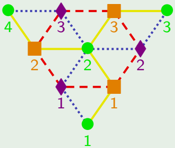
$$A(\mathbf{\Gamma}) = \begin{array}{c c} & \begin{array}{ccc} G & O & P \end{array} \\ \begin{array}{c} G \\ O \\ P \end{array} & \left(\begin{array}{c:c:c} 0 & A^{\mathrm{T}} & C \\ \hdashline A & 0 & B^{\mathrm{T}} \\ \hdashline C^{\mathrm{T}} & B & 0 \end{array}\right) \end{array}$$

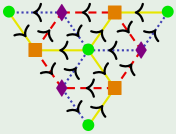
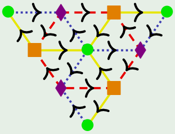
Consider also the oriented adjacency matrices $A(\mathbf{\Gamma}^{\mathrm{X}})$ and $A(\mathbf{\Gamma}^{\mathrm{Y}})$:

$$A(\mathbf{\Gamma}^{\mathrm{X}}) = A(\mathbf{\Gamma}^{\mathrm{Y}})^{\mathrm{T}} = \begin{array}{c c} & \begin{array}{ccc} G & O & P \end{array} \\ \begin{array}{c} G \\ O \\ P \end{array} & \left(\begin{array}{c:c:c} 0 & 0 & C \\ \hdashline A & 0 & 0 \\ \hdashline 0 & B & 0 \end{array}\right) \end{array}$$

Generalized Dynkin diagrams

Example (Type A, Di Francesco-Zuber 1990, Ocneanu 2002)

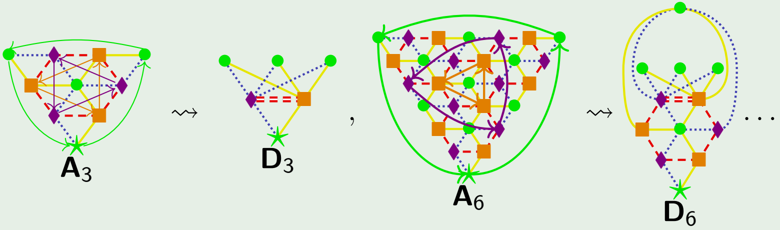
$\mathbf{A}_3 =$ 

, $\mathbf{A}_3^{\mathtt{X}} =$ 
, $\mathbf{A}_3^{\mathtt{Y}} =$ 

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

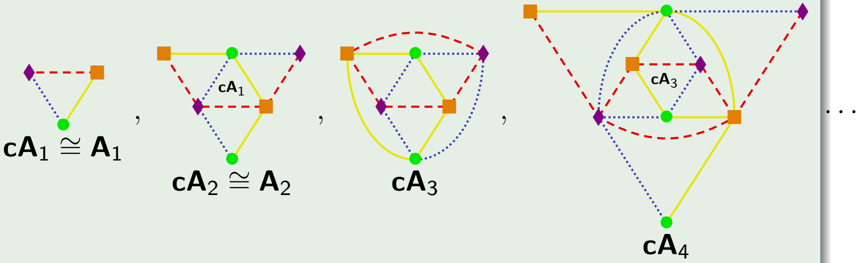
Generalized Dynkin diagrams

Example (Type D, Di Francesco-Zuber 1990, Ocneanu 2002)



Generalized Dynkin diagrams

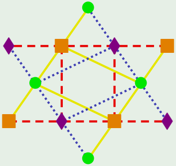
Example (Type cA, Di Francesco-Zuber 1990, Ocneanu 2002)



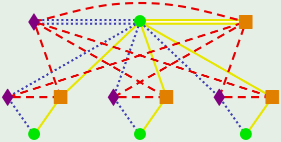
The graph of type $\mathbf{cA}_e$ comes from an iterative procedure on the graph of type $\mathbf{A}_e$.

Generalized Dynkin diagrams

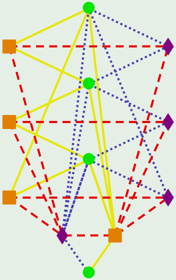
Example (Type E, Di Francesco-Zuber 1990, Ocneanu 2002)



$\mathbf{E}_5$,



$\mathbf{E}_{9_1}$,



$\mathbf{E}_{9_2}$

$+$ three more

Positive integral representations of $\mathrm{H}(I_3(e+3))$

Let Γ be a tricolored generalized ADE Dynkin diagram with generalized Coxeter number $h = e+3$:

Theorem (M-Mazorchuk-Miemietz-Tubbenhauer 2018)

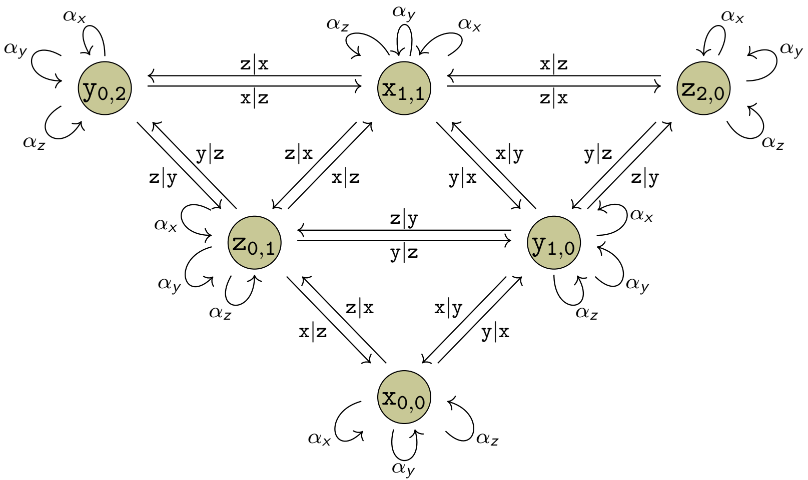
There is a unique (positive) integral $\mathrm{H}(I_3(e+3))$*-representation s.t.*

$$\theta_g \mapsto [2]_{\mathrm{v}} \begin{pmatrix} [3]_{\mathrm{v}}\mathrm{Id} & A^{\mathrm{T}} & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta_o \mapsto [2]_{\mathrm{v}} \begin{pmatrix} 0 & 0 & 0 \\ A & [3]_{\mathrm{v}}\mathrm{Id} & B^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\theta_p \mapsto [2]_{\mathrm{v}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C^{\mathrm{T}} & B & [3]_{\mathrm{v}}\mathrm{Id} \end{pmatrix}.$$

Funct. reps. of $\mathrm{H}(I_3(e+3))$ in generalized type A

Consider the following quiver:



The trihedral zigzag algebra of generalized type A

Definition (M-Mazorchuk-Miemietz-Tubbenhauer 2018)

Let ∇_e be the complex path algebra of Γ modulo the relations:

- Any path with more than one triangle to its left (right) is equal to zero.
- $\alpha_x + \alpha_y + \alpha_z = 0$, $\alpha_x\alpha_y + \alpha_x\alpha_z + \alpha_y\alpha_z = 0$, $\alpha_x\alpha_y\alpha_z = 0$.
- Loops commute with edges.
- $\alpha_z \mathrm{y}|\mathrm{x} = 0$ etc.
- Zig-zag relation: $x|y|x = \alpha_x\alpha_y$ etc.
- Zig-zig equals zag times loop: $x|y|z = \alpha_x x|z$ etc.

The grading on ∇_e is given by twice the path length.

Func. reps. of $\mathrm{H}(I_3(e+3))$ in generalized type A

Let $P_{i,j}$ (resp. ${}_{i,j}P$) be the left (resp. right) graded indecomposable projective ∇_e-module corresponding to vertex $v_{i,j}$ in $\mathbf{\Gamma}$.

Theorem

The assignment

$$\theta_g \mapsto \bigoplus_{i-j\equiv 0 \bmod 3} P_{i,j} \otimes {}_{i,j}P$$

$$\theta_o \mapsto \bigoplus_{i-j\equiv 1 \bmod 3} P_{i,j} \otimes {}_{i,j}P$$

$$\theta_p \mapsto \bigoplus_{i-j\equiv 2 \bmod 3} P_{i,j} \otimes {}_{i,j}P$$

defines a functorial representation of $\mathrm{H}(I_3(e+3))$ *on* $\nabla_e\mathrm{-fpmod}_{gr}$.

Remarks

- By using the $\mathbb{Z}/3\mathbb{Z}$-symmetry on ∇_e, for $e \equiv 0 \bmod 3$, one can easily define the corresponding type D trihedral zigzag algebra. For type cA and E it is not (yet) clear what the right definition is.

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- Unfortunately, we do not know how to lift these functorial representations of $\mathrm{H}(I_3(e+3))$ to full-blown 2-representations of trihedral Soergel bimodules in a straightforward way.

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• Unfortunately, we do not know how to lift these functorial representations of $\mathrm{H}(I_3(e+3))$ to full-blown 2-representations of trihedral Soergel bimodules in a straightforward way.

• There is an alternative construction of simple transitive 2-representations in type A and D, which hopefully works in type cA and E as well. The two approaches are related by the quantum $SU(3)$ McKay correspondence.

The end

THANKS!!!