SHORT- AND LONG-TERM RESEARCH GOALS

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My research at the moment focuses on three particular projects:

▷ 2-representation theory – a subjects which is still in its infant years.
- **Aspects:** Representation theory, categorical algebra, combinatorics.
- **Keywords:** 2-representations, Soergel bimodules, Kazhdan–Lusztig theory, tensor/modular/fusion categories.
- **My latest results:** [MT16], [MMMT16], [MMMT18].
- **Collaborators:** M. Mackaay, V. Mazorchuk and V. Miemietz.

▷ Combinatorial aspects of the representation theory of Lie algebras or finite-dimensional algebras as e.g. diagrammatic presentations or cellularity.
- **Aspects:** Representation theory, combinatorics, low-dimensional topology.
- **Keywords:** Webs, Howe duality, cellular algebras, knot polynomials.
- **My latest results:** [TVW17], [ST17], [ET17].
- **Collaborators:** M. Ehrig, A. Sartori, P. Vaz and P. Wedrich.

▷ Singular topological quantum field theories and their connections to knot theory, Lie theory and geometry.
- **Aspects:** Low-dimensional topology, representation theory, Lie theory.
- **Keywords:** Link homologies, (singular) TQFTs, foams, category $\mathcal{O}$.
- **My latest results:** [EST16], [ETWi16], [ETWe17].
- **Collaborators:** M. Ehrig, C. Stroppel, P. Wedrich and A. Wilbert.

Let me give some details.

1A. 2-representations.

`What?`

Groundbreaking work of Chuang–Rouquier and Khovanov–Lauda on categorifications of quantum groups and their representations opened a completely new field of research which one could call 2-*representation theory*.

However, their works were mostly example-based and a general theory of 2-representations was missing. Working in this direction, Mazorchuk–Miemietz came up with a theory which plays the 2-categorical role of the theory of representation theory of finite-dimensional algebras. Let us call this theory *finitary* 2-*representation theory*, with the finitary referring to some finiteness conditions. As an example, while representation theory studies the representations of Hecke algebras of Coxeter groups, finitary 2-representation theory would study the 2-representations of the categorification of these Hecke algebras called Soergel bimodules.

`My motivation.`

The subject of finitary 2-representation theory is rapidity growing with several new results published every year, but it is still widely open at the moment, and a lot of natural and interesting questions remain open.

Since I believe in the future usefulness of Mazorchuk–Miemietz's approach (think e.g. about the impact of representation theory in other fields of mathematics – the same should be true on the 2-categorical level), I want continue my research on 2-representations of Soergel bimodules and related 2-categories.

`My latest results.`

In [MT16] we classify the simple 2-representations of the dihedral Soergel bimodules, which we see as a starting point of studying more general Soergel bimodules. Note that the approach taken in [MT16] is motivated by joint work with Andersen. In this joint work we basically connect the Soergel bimodules in dihedral type to the representation theory of quantum $\mathfrak{sl}_2$ at roots of unity by defining a 2-representation on the category of tilting modules of quantum $\mathfrak{sl}_2$. Thus, one can expect a direct connection of our work to the work of Kirillov–Ostrik on "finite subgroups" of quantum $\mathfrak{sl}_2$ at roots of unity and the quantum analogue of the McKay correspondence (with potential connection to modular invariants in conformal field theory). This connection is made rigorous in joint work with Mackaay–Mazorchuk–Miemietz [MMMT16]. Morally: "finite subgroups" of quantum $\mathfrak{sl}_2$ at roots of unity are "fusion subquotients" of the 2-category of singular Soergel bimodules of dihedral type. A generalization of this to higher ranks is non-trivial, but was worked out in [MMMT18] with a lot of interesting questions remaining open.

`Next steps?`

The following are explicit questions on which I work jointly with (a subset of) Mackaay, Mazorchuk and Miemietz.

Since the ideas from the joint work with Andersen heavily draw from a construction of Khovanov–Seidel (which is related to categorical actions of braid groups), one might expect connections from [MT16] to braid groups. Another path we are exploring is that we hope that the construction presented in [MT16] generalizes to other e.g. (infinite) Coxeter types.

Moroever, in [MMMT18] we defined a new algebras and its categorification and even some basic questions of this are still open.

Finally, in [MMMT16] we made some connections of finitary 2-representation theory to the theory of tensor categories, but in an example. We are trying to develop the abstract 2-representation theory further and hope to make some connections to this broad field on a more general level.

1B. Diagrammtic representation theory.

`What?`

Consider the following question: Given some Lie algebra, can one give a generator-relation presentation for the category of its finite-dimensional representations, or for some well-behaved subcategory?

Maybe the best-known instance of this is the case of the monoidal category generated by the vector representation of $\mathfrak{sl}_2$. Its generator-relation presentation is known as the Temperley–Lieb category and goes back to work of Rumer–Teller–Weyl and Temperley–Lieb (the latter in the quantum setting). Note that the Temperley–Lieb category is given diagrammatically which eases to work with it, and the relation from it to representations of $\mathfrak{sl}_2$ is basically given by the classical Schur–Weyl duality.

`My motivation.`

Apart from some well-behaved cases, finding suitable generators and relations is a very difficult task. However, the known examples are very interesting, provide some nice algebras (as endomorphism algebras of the studied categories) and related to many parts of modern mathematics and physics. Most prominent example of such algebras are Temperley–Lieb algebras, Brauer algebras, Hecke algebras, arc algebras, web algebras, (cyclotomic) KL–R algebras and many others.

Note that all of these are given diagrammatically which comes with the upshot that connections to low-dimensional topology become evident. For example, the Temperley–Lieb algebras give a way to define and compute the celebrated Jones polynomial of a knot.

`My latest results.`

In [TVW17] we use quantum super Howe duality – which is a beefed-up version of Schur–Weyl duality – to define and study module categories for quantum $\mathfrak{gl}_{N|M}$. This has the upshot that we can define the HOMFLY–PT polynomial of links using our calculus, and we can show several properties and symmetries of this polynomial. Moreover, we hope that this is categorifyable and will be helpful to prove the corresponding symmetries for the associated link homologies.

The situation studied in [TVW17] is in Lie type A – as almost all known results in this field are in Lie type A. In [ST17] we go outside of this Lie type (where basically nothing is known about generators and relations) and obtain not just a quantization of type BCD Howe dualities, but also a generalization of Brauer's celebrates algebra.

In [ET17] we go into a different direction and study the representation theory of such diagram algebras. More precisely, we generalize the notion of cellular algebras to what we call relative cellular algebras and develop a theory of such algebras. We also give several examples which are relative cellular, but not cellular.

`Next steps?`

The following are explicit questions which remain open.

The quantization of Howe duality in [ST17] features a so-called coideal subalgebra together with a quantum group, and is asymmetric. This is a hint that there is also an alternative way to do it where the coideal subalgebra and the quantum group swap roles. This should lead to new way to study link polynomials outside of type A. Moreover, a categorification of this story is completely mysterious at present, and its a worthwhile goal to be studied.

Finally, in [ET17] most of our example are related to characteristic p representation theory. It is a striking question whether this is just a coincidence or whether there is some deeper connection between the theories.

1C. **Singular topological quantum field theories.**

`What?`

In his pioneering work, Khovanov introduced the so-called arc algebra. One of his main purposes was to extend his celebrated categorification of the Jones polynomial from links to tangles. His idea was to interpret the link homology as certain bimodules of the arc algebra.

One of Khovanov's main ideas (as developed further by Bar-Natan) was that the arc algebras are obtained via 1+1-dimensional topological quantum fields theories (TQFTs) as originate in work of physicists and axiomatized by Atiyah–Segal in the end of the 1980ties. Note hereby that the formulation by Khovanov–Bar-Natan can be seen as the $\mathfrak{sl}_2$ case of a broader story, where the generalization of the usage of 1+1-dimensional TQFTs is replaced by what is called a singular 1+1-dimensional TQFTs.

`My motivation.`

These TQFTs describe the cobordisms between the links and tangles and have led to several new results in smooth four-dimensional topology, e.g. a combinatorial proof of the Milnor conjecture regarding slice torus knots by Rasmussen and purely combinatorial constructions of exotic structures in four space by Gompf–Rasmussen.

This series of results has led to several variations and generalizations of Khovanov's original formulation, utilized in a large body of work by several researchers.

For example, left aside its knot theoretical origin, the arc algebra has interesting representation theoretical, algebraic geometrical and combinatorial properties. For instance, there are relations to (cyclotomic) KL–R algebras, knot homologies, to the Alexander polynomial and knot Floer homology, to the representation theory of Brauer's centralizer algebras and to Lie superalgebras – just to name a few.

`My latest results.`

In [EST16] we quantize the ideas Khovanov–Bar-Natan and obtain a version of singular TQFTs with many parameters. Some of these describe Khovanov homology, some its functorial cousin. We show however that all of these different parameter versions are equivalent, showing that Khovanov homology and its functorial cousin are also equivalent.

In joint work with Ehrig–Wilbert [ETWi16], we constructed the type D version of Khovanov's arc algebra using a singular TQFT approach à la Khovanov–Bar-Natan, revealing its potential application in low-dimensional topology. Note hereby that (again) only Lie type A is well-understood and any steps outsides of type A are novel and interesting.

Finally, in [ETWe17] we prove that the $\mathfrak{sl}_N$ version of Khovanov homology is functorial – a question which was believed to be true but remained opened for about 10 years.

`Next steps?`

There is one particular striking question which remains open:

While the representation theoretical origin, connections and properties of the type D version of Khovanov's arc algebra are fairly well-understood, its connection to low-dimensional topology remains mysterious.

This is a question which I am currently investigating together with Stroppel–Wilbert, and we have some ideas about potential link homologies for links in certain orbifolds, generalizing classical links.

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