LECTURE: QUANTUM TOPOLOGY WITHOUT TOPOLOGY

What?
Quantum invariants are more than just topological invariants needed to tell objects apart. They build bridges between topology, algebra, number theory and quantum physics helping to transfer ideas, and stimulating mutual development. They also have a deep and interesting connection to representation theory, in particular, to representations of quantum groups.

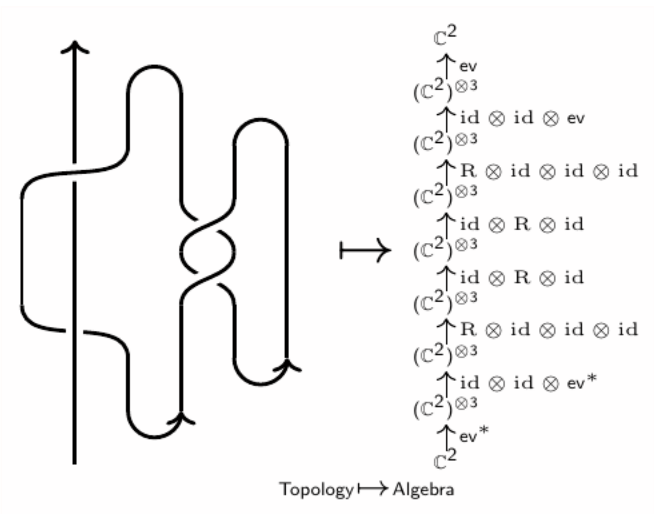
In this course we will introduce these objects from different perspectives: skein and representation theoretic. We will start with the Jones polynomial, study its properties, and then move to the categorification of this polynomial discovered by Khovanov. In the second part (i.e. this lecture) of the class we will explain its connections to representation theory following the ideas of e.g. Reshetikhin–Turaev, and then explain how the categorification also arises from very natural constructions in categorical representation theory.

The lecture follows several texts, e.g. [BS11], [EGNO], [HV19] or [TV17].

Who?
MSC or PhD students in Mathematics interested in a mixture of (linear) algebra, topology and category theory, but everyone is welcome.

Where and when?

- ▶ Time and date.
  - Every Monday from 10:15–12:00.
  - Online, see https://www.youtube.com/channel/UC7wuTzExlC6RDhCjmNdd4Hw.
  - First lecture: Monday 01.Feb.2021. Last lecture: Monday 05.Apr.2021.
- ▶ Website http://www.dtubbenhauer.com/lecture-qinv-2021.html



Schedule and some details.

▷ 1^{th} talk "Categories – definitions, examples and graphical calculus".

- **Speaker.** Daniel Tubbenhauer.
- **Date.** 01.Feb.2021, 10:15–12:00.
- **Topic.** A primer on categories.
- **Plan.** After explaining the whole concept of the lecture, we start by introducing categories, functors and natural transformations. Then the graphical calculus for categories is introduced (called string, Penrose or Feynman diagrams). The whole story is wrapped-up by giving many examples, in particular, the category of (unoriented and oriented) tangles.
- **Main goals.** To explain why categories are 1-dimensional objects.
- **Note.** Category theory lays the focus on the morphisms. Still, most categories are named after their objects.
- **Literature.** Mostly the corresponding sections in [BS11] and [TV17].

▷ 2^{th} talk "Monoidal categories I – definitions, examples and graphical calculus".

- **Speaker.** Daniel Tubbenhauer.
- **Date.** 08.Feb.2021, 10:15–12:00.
- **Topic.** Strictification.
- **Plan.** We start by introducing monoidal categories and monoidal functors. After giving many example, in particular, of course, the monoidal category of tangles, we present the coherence and strictification theorem for monoidal categories. At the very end a first instance of the monoidal graphical calculus.
- **Main goals.** The coherence and strictification theorem. We also sketch why monoidal categories are 2-dimensional objects.
- **Note.** It should be clear after this lecture that the category of tangles is the main example for this course.
- **Literature.** Mostly the corresponding sections in [BS11] and [TV17].

▷ 3^{th} talk "Monoidal categories II – more graphical calculus".

- **Speaker.** Daniel Tubbenhauer.
- **Date.** 15.Feb.2021, 10:15–12:00.
- **Topic.** Generators (and relations).
- **Plan.** After recalling and introducing more graphical calculus, we discuss how monoidal categories can be generated, sometimes (meaning almost always) with relations. Note that generators–relations for (any kind of) categories are very similar to the same concept for groups.
- **Main goals.** To explain why monoidal categories are 2-dimensional objects, and monoidal generators.
- **Note.** "Free equals easy".

- **Literature.** Mostly the corresponding sections in [BS11] and [TV17].

▷ 4th talk "Pivotal categories – definitions, examples and graphical calculus".

- **Speaker.** Daniel Tubbenhauer.
- **Date.** 22.Feb.2021, 10:15–12:00.
- **Topic.** Traces and dimensions.
- **Plan.** Similarly as before, we start by introducing pivotal categories and pivotal functors and their graphical calculus; the category of tangles being the prototypical example. Then it is explained how one can generalize the notions of traces and dimensions in these categories.
- **Main goals.** Explain the concept of categorical traces and categorical dimensions.
- **Note.** In the category of tangles the endomorphisms of the unit are non-trivial as they are all links.
- **Literature.** Mostly the corresponding sections in [TV17].

▷ 5th talk "Braided categories – definitions, examples and graphical calculus".

- **Speaker.** Daniel Tubbenhauer.
- **Date.** 01.Mar.2021, 10:15–12:00.
- **Topic.** Braids and categories.
- **Plan.** Clearly, we start by introducing braided categories and braided functors. After giving many example, in particular, of course, the braided category of tangles, we present the Reidemeister calculus. The prototypical example—the category of braids—will be discussed.
- **Main goals.** We sketch why braided categories are 2.5-dimensional objects by introducing the Reidemeister calculus.
- **Note.** A braiding is an additional choice of data—a monoidal category can have many braidings.
- **Literature.** Mostly the corresponding sections in [BS11] and [TV17].

▷ 6th talk "Additive, linear and abelian categories – definitions and examples".

- **Speaker.** Daniel Tubbenhauer.
- **Date.** 08.Mar.2021, 10:15–12:00.
- **Topic.** Enter: Linear algebra.
- **Plan.** After introducing the notions of additive, linear and abelian categories, we discuss several notions and properties such as exact sequences, projective and injective objects etc.
- **Main goals.** To explain the main concepts of additive categories, generalizing linear algebra, and abelian categories, generalizing homological algebra.
- **Note.** The category of tangles is neither additive nor linear nor abelian, but their main target categories are.

- **Literature.** From now on we start using [EGNO].

▷ 7th talk "Fiat and tensor categories – enrich the concepts from before".
- **Speaker.** Daniel Tubbenhauer.
- **Date.** 15.Mar.2021, 10:15–12:00.
- **Topic.** Functors with structure.
- **Plan.** Explain how various concepts such as monoidal functors can and should be linearized in the case the corresponding categories have more structure.
- **Main goals.** To explain how to linearize the lectures 1–5.
- **Note.** Functors between categories with additional structure should have additional properties.
- **Literature.** Mostly the corresponding sections in [EGNO].

▷ 8th talk "Fiat, tensor and fusion categories – definitions and classifications".
- **Speaker.** Daniel Tubbenhauer.
- **Date.** 22.Mar.2021, 10:15–12:00.
- **Topic.** Fusion and graphical calculus.
- **Plan.** After introducing the notion of fusion categories, we will discuss some properties of these such as that they are non-degenerate. Then we explain how to beef-up the usual calculus to the case of fusion categories.
- **Main goals.** More diagrammatics.
- **Note.** One can say quite a lot about fusion categories as they are generalizations of groups.
- **Literature.** Mostly the corresponding sections in [EGNO] and [TV17].

▷ 9th talk "Fusion and modular categories – definitions and graphical calculus".
- **Speaker.** Daniel Tubbenhauer.
- **Date.** 29.Mar.2021, 10:15–12:00.
- **Topic.** Modular categories.
- **Plan.** The whole point of this lecture is to explain three classes of examples and some of their properties in detail. These classes are representations of a finite group G, dually, G graded vector spaces and also fusion categories coming from quantum groups. Some of them are modular, in particular, the so-called quantum doubles which are discussed.
- **Main goals.** To give examples.
- **Note.** Modular categories are pretty hard to construct.
- **Literature.** Mostly the corresponding sections in [EGNO] and [TV17], but also [BK01].

▷ 10^{th} talk "Quantum invariants – a diagrammatic approach".

- **Speaker.** Daniel Tubbenhauer.
- **Date.** 05.Apr.2021, 10:15–12:00.
- **Topic.** Witten–Reshetikhin–Turaev.
- **Plan.** We finally explain the main idea behind quantum invariants, i.e. functors from the category of tangles to suitable target categories.
- **Main goals.** To wrap-up the class.
- **Note.** Because we assume no prior knowledge about low-dimensional topology, there are no details given.
- **Literature.** Pretty much open and a bit sketchy, but I use also [BK01].

References

[BS11] J. Baez, M. Stay. Physics, Topology, Logic and Computation: A Rosetta Stone. New structures for physics, 95–172, Lecture Notes in Phys., **813**, Springer, Heidelberg, 2011. https://arxiv.org/abs/0903.0340

[BK01] B. Bakalov, A. Kirillov Jr. Lectures on tensor categories and modular functors. University Lecture Series, **21**. American Mathematical Society, Providence, RI, 2001.

[EGNO] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik. Tensor categories. Mathematical Surveys and Monographs **205**. American Mathematical Society, Providence, RI, 2015. http://www-math.mit.edu/~etingof/egnobookfinal.pdf

[HV19] C. Heunen, J. Vicary. Categories for Quantum Theory: An Introduction. Oxford Graduate Texts in Mathematics, **28**. Oxford University Press, Oxford, 2019. xii+336 pp. Comes close: http://www.cs.ox.ac.uk/people/jamie.vicary/IntroductionToCategoricalQuantumMechanics.pdf

[TV17] V. Turaev, A. Virelizier. Monoidal categories and topological field theory. Progress in Mathematics, **322**. Birkhäuser/Springer, Cham, 2017.

Daniel Tubbenhauer: Fellow, Hausdorff Center for Mathematics, Villa Maria, Endenicher Allee 62, D-53115 Bonn, Germany, www.dtubbenhauer.com

Email address: dtubbenhauer@gmail.com