MAT564: SEMINAR REPRESENTATION THEORY OF $\mathfrak{sl}_2$ – PLAN

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Disclaimer

Please do not hesitate to contact me in case of questions. (The contact data can be found below.) Never forget to give many examples during your talk!

What?

Representation theory is an important and intensively studied area of modern mathematics with applications to basically all major areas of mathematics and physics.

The aim of this seminar is to learn what representation theory is all about, with the focus on the toy example of $\mathfrak{sl}_2$ where everything can be done explicitly.

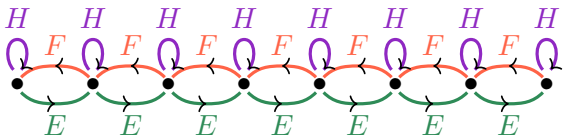
The seminar follows the book [Ma10].

Who?

Bachelor students, Master students and upwards interested in a mixture of algebra and category theory. In particular, students following the lecture "Introduction to representation theory" by Anna Beliakova. (This seminar will start mid of March after the first concepts were introduced in the lecture "Introduction to representation theory".)

Where and when?

- Time and date.
  - Every Monday from 10:15–12:00.
  - Room Y27H28, University Zurich, Institute of Mathematics.
  - First meeting: Monday 18.Mar.2019.
- Preliminary meeting: Friday 01.Feb.2019, 10:15–12:00, room Y27H28.
- Website http://www.dtubbenhauer.com/seminar-sl2-2019.html



Schedule and some details.

- 1^{th} talk "The finite-dimensional case I – the simples".
  - **Speaker.** Mariya.
  - **Date.** 18.Mar.2019, 10:15–12:00.

- **Topic.** $\mathfrak{sl}_2$, $\mathfrak{sl}_2$-modules and classification of all finite-dimensional simple $\mathfrak{sl}_2$-modules.
- **Plan.** Start with the definition of the Lie algebra $\mathfrak{sl}_2$ and its first basic properties [Ma10, Exercises 1.1.1 and 1.1.3, and Lemmas 1.1.2 and 1.1.7]. Then define what a $\mathfrak{sl}_2$-module is a give the first main examples [Ma10, Examples 1.1.4 to 1.1.6]. Then define what a map between $\mathfrak{sl}_2$-modules, what a submodule and a simple module is. Give examples along the way [Ma10, Remaining Exercises, Examples and Lemmas in Section 1.1]. Prepare the stage for the main theorem of the talk [Ma10, Lemma 1.2.1 to Exercise 1.2.7], and present the classification [Ma10, Theorem 1.2.8]. Finally, give the matrix representation of the operators E, F, H.
- **Main goals.** The proof of [Ma10, Theorem 1.2.8].
- **Note.** [Ma10, (1.9)] is the main picture of the book. Explain it carefully. Do not ignore the exercises, *e.g.* [Ma10, Exercise 1.2.7] is very important.
- **Literature.** [Ma10, Sections 1.1 and 1.2].

▷ 2$^{\text{th}}$ talk "The finite-dimensional case II – semisimplicity".

- **Speaker.** Samuel.
- **Date.** 25.Mar.2019, 10:15–12:00.
- **Topic.** Finite-dimensional $\mathfrak{sl}_2$-modules are semisimple.
- **Plan.** Start with the main statement of the talk, *i.e.* [Ma10, Theorem 1.3.3], where you first need to introduce some notations. Then you need to prepare the stage to prove this theorem, where you should start with an important tool, *i.e.* [Ma10, Theorem 1.2.12]. Another important tool is the Casimir operator and its basic properties [Ma10, Lemma 1.3.4 to Exercise 1.3.7]. Then you are ready to prove the theorem and an important consequence [Ma10, Corollary 1.3.8.]. Give the decomposition of tensor products [Ma10, Section 1.4] as an application.
- **Main goals.** Explain the proof of [Ma10, Theorem 1.2.12] carefully.
- **Note.** If you run out of time, then keep the proof of [Ma10, Theorem 1.4.5] short.
- **Literature.** [Ma10, Sections 1.2 to 1.4].

▷ 3$^{\text{th}}$ talk "The finite-dimensional case III – unitarizability".

- **Speaker.** Arno.
- **Date.** 01.Apr.2019, 10:15–12:00.
- **Topic.** The concept of adjointness of E and F, and characters.
- **Plan.** Introduce extra structure on a $\mathfrak{sl}_2$-module called the *- and the $^\diamond$-operation and prove that any finite-dimensional $\mathfrak{sl}_2$-module has such extra structures [Ma10, Theorems 1.5.1 and 1.5.5]. Explain why both structures are basically unique for simple modules [Ma10, Propositions 1.5.6 and 1.5.7]. (The same holds for the $^\natural$-operation define in [Ma10, Exercise 1.8.16].) Only briefly discuss the case over the real numbers, but rather present the concept of a character, *i.e.* [Ma10, Exercise 1.8.22], which should remind the audience of the case of finite groups.
- **Main goals.** Explain unitarizability and characters.

- **Note.** Unitarizability is an important concept (which you might recognize from the representation theory of finite groups) and basically boils down to E and F being adjoint operators.
- **Literature.** [Ma10, Sections 1.5 to 1.6].

Easter break.

▷ 4th talk "Universal enveloping algebra I – the PBW theorem".

- **Speaker.** Mariya.
- **Date.** 08.Apr.2019, 10:15–12:00.
- **Topic.** Basic properties of $\mathrm{U}(\mathfrak{sl}_2)$.
- **Plan.** Introduce the concept of the universal enveloping algebra and some of its basic properties as *e.g.* [Ma10, Theorem 2.1.5] or [Ma10, Proposition 2.1.6]. The main statement is then [Ma10, Corollary 2.1.9], which needs to be explained carefully. Finally state and prove [Ma10, Theorem 2.2.1], which will occupy the remaining time of the talk.
- **Main goals.** Explain the PBW theorem [Ma10, Theorem 2.2.1] and its proof.
- **Note.** [Ma10, Remark 2.2.10] is very important and can not be stressed often enough. Stress also [Ma10, Corollary 2.2.7].
- **Literature.** [Ma10, Sections 2.1 to 2.2].

▷ 5th talk "Universal enveloping algebra II – the Cartan subalgebra".

- **Speaker.** Daniel.
- **Date.** 15.Apr.2019, 10:15–12:00.
- **Topic.** More basic properties of $\mathrm{U}(\mathfrak{sl}_2)$.
- **Plan.** Start by explaining how the PBW theorem gives a filtration on $\mathrm{U}(\mathfrak{sl}_2)$ whose associated graded is just a polynomial algebra. (This is basically the content of [Ma10, Section 2.3].) The next goal is to explain what the Casimir element is, and its importance, *i.e.* [Ma10, Theorem 2.4.5]. Also explain how the Casimir and H span the degree 0 part [Ma10, Proposition 2.4.4].
- **Main goals.** Explain what the center of $\mathrm{U}(\mathfrak{sl}_2)$ [Ma10, Theorem 2.4.5] is and how it simplifies the PBW statement [Ma10, Theorem 2.4.6].
- **Note.** [Ma10, Exercise 2.8.12] is very nice and should be presented.
- **Literature.** [Ma10, Sections 2.3 to 2.4].

Easter break.

▷ 6th talk "Universal enveloping algebra III – the Harish-Chandra homomorphism".

- **Speaker.** Mariya.
- **Date.** 29.Apr.2019, 10:15–12:00.

- **Topic.** Further properties of $\mathrm{U}(\mathfrak{sl}_2)$.
- **Plan.** Introduce the map κ, and show and explain some of its main properties. Hereby [Ma10, Proposition 2.5.2] and [Ma10, Corollary 2.5.4] should be explained very carefully since they are important for the general theory. Then finish this part of the seminar with [Ma10, Theorem 2.6.2].
- **Main goals.** Explain the important idea of the Harish-Chandra homomorphism and prove a technical, yet crucial, property of $\mathrm{U}(\mathfrak{sl}_2)$ [Ma10, Theorem 2.6.2].
- **Note.** You should have enough time to also discuss [Ma10, Exercise 2.8.13].
- **Literature.** [Ma10, Sections 2.5 to 2.6].

▷ 7^{th} talk "Weight modules I – weight, Verma and dense modules".

- **Speaker.** Daniel.
- **Date.** 06.May.2019, 10:15–12:00.
- **Topic.** The ideas of Verma in case of $\mathfrak{sl}_2$.
- **Plan.** Introduce the concept of a weight module (this is [Ma10, Section 3.1]) with [Ma10, Corollary 3.1.7] being the crucial statement of the section. The first examples of weigh modules are the finite-dimensional modules and this should be stressed. The second example of weight modules are Verma modules, which you then should introduce. (This is [Ma10, Section 3.2].) The crucial statement is then [Ma10, Corollary 3.2.6] which needs to be explained carefully. Finally, briefly discuss another example, *i.e.* the so-called dense modules as in [Ma10, Section 3.3].
- **Main goals.** Explain the ideas of Verma modules and how they serve as a way to get the finite-dimensional simples.
- **Note.** You have a lot of material. so you need to be brief at some points. Focus on the main statements, and skip proofs if necessary.
- **Literature.** [Ma10, Sections 3.1 to 3.3].

▷ 8^{th} talk "Weight modules II – the simples".

- **Speaker.** Mariya.
- **Date.** 13.May.2019, 10:15–12:00.
- **Topic.** The classification of all simple weight modules.
- **Plan.** The first part of your talk is devoted to prove [Ma10, Theorem 3.4.1]. Hereby also do [Ma10, Exercises 3.4.4 to 3.4.5] at the end of the corresponding section. Take your time
- **Main goals.** Explain [Ma10, Theorem 3.4.1].
- **Note.** Take your time to prove [Ma10, Theorem 3.4.1], and see how far you can get with [Ma10, Section 3.5].
- **Literature.** [Ma10, Sections 3.4 to 3.5].

▷ 9th talk "Weight modules III – categorical considerations".

- **Speaker.** Daniel.
- **Date.** 20.May.2019, 10:15–12:00.
- **Topic.** Categories of $\mathfrak{sl}_2$-modules.
- **Plan.** This talk is very much open – start where the previous speaker left and try to get as far as possible with the material, carefully omitting same statements and also proofs. The crucial statements are the explicit characterizations of the categories of the corresponding weight modules, *i.e.* [Ma10, Theorems 3.7.3, 3.8.6 and 3.9.5].
- **Main goals.** The main statements which I would like to see are the classification theorems [Ma10, Theorems 3.7.3, 3.8.6 and 3.9.5].
- **Note.** This talk is certainly the 'hardest' one of the seminar.
- **Literature.** [Ma10, Sections 3.6 to 3.9].

▷ 10th talk "Outlook – category $\mathcal{O}$".

- **Speaker.** Mariya.
- **Date.** 27.May.2019, 10:15–12:00.
- **Topic.** The (in)famous category $\mathcal{O}$.
- **Plan.** Define a very specific category of $\mathfrak{sl}_2$-weight modules called category $\mathcal{O}$, and show its first basic properties. (This is [Ma10, Sections 5.1 and 5.2].) What is not obvious from the definition of $\mathcal{O}$ is its importance in representation theory, stress this. Main examples of modules in $\mathcal{O}$ are finite-dimensional and Verma modules. In particular, [Ma10, Corollary 5.2.8] is very important and needs to be presented. Finish by carefully preparing the stage to present the main statement of your talk which is [Ma10, Theorem 5.3.1].
- **Main goals.** The explicit description of $\mathcal{O}$ in [Ma10, Theorem 5.3.1].
- **Note.** You have a lot of non-trivial material to cover. Be as brief as possible and sacrifice proofs if necessary.
- **Literature.** [Ma10, Sections 5.1 to 5.4].

References

[Ma10] V. Mazorchuk. *Lectures on $\mathfrak{sl}_2(\mathbb{C})$-modules*. Imperial College Press, London, 2010. x+263 pp.

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