RESEARCH STATEMENT

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ABSTRACT. This is my research statement for the year 2016 containing some introductory words for the my most recent research questions and projects (finished or still work in progress). Throughout: My intention is not to give rigorous mathematical definitions or statements, but rather an informal overview about my research. (I hope the reader will forgive me some of my sloppy formulations.)

In short: My main research interest is 2-categorical representation theory (of e.g. "categorified" Coxeter groups), categorification (of e.g. quantum groups) and its applications in representation theory, low-dimensional topology and algebraic geometry. In particular, I am interested in algebraic, combinatorial and diagrammatic aspects of categorification. I am also interested in related topics as for example representation theoretical questions about Hecke/Brauer algebras or Lie groups and modular representation theory.

Indeed, my research interest at the moment basically splits into a "topologically motivated" part concerning algebraic constructions of link homologies and their functoriality, see Subsection 2.1, as well as a "representation theory motivated" part concerning "higher" representation of Coxeter groups, see Subsection 2.2.

But before going into the details of my own current research, let me try to motivate the basic questions which play a major role in all of my research interests.

1. Introduction

1.1. Categorification.

Seeking "higher" structure. The notion categorification was introduced by Crane in [18] based on an earlier joint work with Frenkel [19]. But the concept of categorification has a much longer history than the word itself. Forced to explain the concept in one sentence, I would choose

"Interesting integers are shadows of richer structures in categories."

The basic idea can be seen as follows. Take a "set-based" structure S and try to find a "category-based" structure $\mathbf{C}$ such that S is just a shadow of the category $\mathbf{C}$. If the category $\mathbf{C}$ is chosen in a "good" way, then one has an explanation of facts about the structure S in a categorical language. That is, certain facts in S can be explained as special instances of natural constructions.

Experience tells us that the categorical structure does not only explain properties of the "set-based" structure, but is usually a much richer and more interesting.

In principal, one can perform such a "categorification process" on any level, e.g. one can categorify an "n-category like structure" into an "$n{+}1$-category like structure". Without going into any details about higher categories, the slogan for me is that a set is a collection of "number like structures" with the set of natural numbers as the blueprint example; a category is a collection of "set like structures" with the category of sets as a blueprint example; a 2-category is a collection of "category like structures" with the 2-category of categories as a blueprint example; a 3-category is a collection of "2-category like structures"...

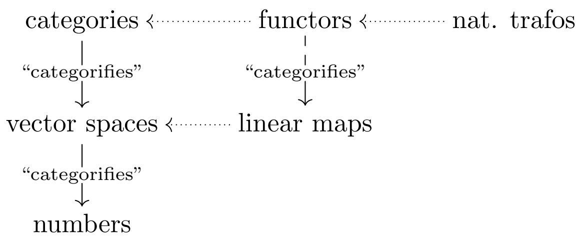
Decategorification. Categorification comes with an "inverse" called decategorification; and categorification can be seen as "remembering" or "inventing" information while decategorification is more like "forgetting" or "identifying" structure, which is, of course, easier.

We usually have to specify what we mean by decategorification. A blueprint example is that the category of $\mathbb{K}$-vector spaces $\mathbb{K}\mathbf{Vect}$ (over some field $\mathbb{K}$) "categorifies" the natural numbers $\mathbb{Z}_{\geq 0}$. In this case the decategorification is given by taking dimensions.

Another "standard" example is to take (a suitable version of) the Grothendieck group $[\cdot]_{\mathbb{K}} = K_0(\cdot) \otimes_{\mathbb{Z}} \mathbb{K}$. (For example $K_0(\mathbb{K}\mathbf{Vect}) \cong \mathbb{Z}$ and $[\mathbb{K}\mathbf{Vect}]_{\mathbb{K}} \cong \mathbb{K}$.)

In fact, if we think of the category $\mathbb{K}\mathbf{Vect}$ as being a "set-based" structure (with objects being sets with a $\mathbb{K}$-linear structure), then we might want to categorify this further by considering the 2-category $\mathbb{K}\mathfrak{Cat}$ of $\mathbb{K}$-linear categories, $\mathbb{K}$-linear functors and $\mathbb{K}$-linear natural transformations. Taking an appropriate 2-categorical Grothendieck group recovers $\mathbb{K}\mathbf{Vect}$.

One diagram is worth a thousand words. Each step of a "categorification process" should reveal more structure. An illustration for the example from above is the following (omitting the $\mathbb{K}$):



Here we first "categorify" numbers into $\mathbb{K}$-vector spaces. The new information available are now $\mathbb{K}$-linear maps between $\mathbb{K}$-vector spaces. (Thus, we have the whole power of linear algebra at hand.) There is no reason to stop: we can "categorify" $\mathbb{K}$-vector spaces into $\mathbb{K}$-categories, $\mathbb{K}$-linear maps into $\mathbb{K}$-linear functors. Again, we see a new layer of information, namely the natural transformations between these functors.

Examples of categorification. The following list of example is already long, but biased and far from being complete. Much more can be found in the work of Baez and Dolan [4] and [5] for examples that are related to more combinatorial parts of categorification or Crane and Yetter [20], Khovanov, Mazorchuk and Stroppel [36], Mazorchuk [50] or Savage [65] for examples from algebraic categorification.

- ▷ In some sense the "most classical, but quite recent" example is Khovanov's categorification of the Jones (or $\mathfrak{sl}_2$) polynomial [32].
- ▷ Khovanov's construction can be extended to a categorification of the Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial and the HOMFLY-PT polynomial, e.g. see [39]. Moreover, some "applications" of Khovanov's categorification are:
  - ▶ It is functorial, e.g. see [15], [17] or [9]: it "knows" about link cobordisms. Since such cobordisms are cobordisms embedded in the four-space, this gives a way to get information about smooth structures in dimension 4. (And 4-dimensional, smooth topology is hell!)

- ▶ Kronheimer and Mrowka showed in [43], by comparing Khovanov homology to Knot Floer homology, that Khovanov homology detects the unknot. This is still an open question for the Jones polynomial.
- ▶ Rasmussen obtained his famous invariant by comparing Khovanov homology to a variation of it. He used his invariant to give a combinatorial proof of the Milnor conjecture, see [58]. Note that he also gives in [59] a way to construct exotic $\mathbb{R}^4$ from his approach.
- ▶ There is a variant of Khovanov homology, called odd Khovanov homology, see [56], that differs over $\mathbb{Q}$ and can not be seen on the level of polynomials.
- ▶ There is a variant that categorifies the HOMFLY-PT polynomial. This categorification is a "rich" structure itself and has a lot of connections to various parts of mathematics and related fields, see e.g. [30] and the references therein.
- ▶ Not the main point but: it is strictly stronger than the Jones polynomial.

▷ Other notable categorifications related to low-dimensional topology are:

- ▶ Floer homology can be seen as a categorification of the Casson invariant of a manifold. Floer homology is "better" than the Casson invariant, e.g. it is possible to construct a 3+1 dimensional TQFT which for closed four-dimensional manifolds gives Donaldson's invariants, see for example [74].
- ▶ Knot Floer homology can be seen as a categorification of the classical knot invariant of Alexander and Conway, see for example [57].
- ▶ The approach to categorify the Reshetikhin-Turaev $\mathfrak{g}$-polynomial for arbitrary simple Lie algebra $\mathfrak{g}$ by Webster [72].

▷ The notion categorification is from the interplay of low-dimensional topology and representation theory. Hence, there are also several examples coming from representation theory as e.g.:

- ▶ Ariki gave in [3] a remarkable categorification of all finite-dimensional, irreducible representation of $\mathfrak{sl}_n$ for all n as well as a categorification of integrable, irreducible representations of the affine version $\widehat{\mathfrak{sl}}_n$. In short, he identified the Grothendieck group of blocks of so-called Ariki-Koike cyclotomic Hecke algebras with weight spaces of such representations in such a way that direct summands of induction and restriction functors between cyclotomic Hecke algebras for $m, m+1$ act on the K_0 as the E_i, F_i of $\mathfrak{sl}_n$.
- ▶ Chuang and Rouquier masterfully used in [16] the categorification of good old $\mathfrak{sl}_2$ to solve an open problem in modular representation theory of the symmetric group using a completely new approach.
- ▶ Khovanov and Lauda [35], and independently Rouquier [63] have categorified all quantum Kac-Moody algebras with their canonical bases.
- ▶ Khovanov and Qi [38] and Elias and Qi [27] have an approach how to categorify at roots of unity. Their categorification of quantum $\mathfrak{sl}_2$ for the quantum parameter q being a (certain type of) root of unity can be (the future will prove me right or wrong) the first step to categorify the Witten-Reshetikhin-Turaev invariants of 3-manifolds.
- ▶ The so-called Soergel category $\mathfrak{S}$ can be seen in the same vein as a categorification of the Hecke algebras in the sense that the split Grothendieck group gives the Hecke algebras. We note that Soergel's construction shows

that Kazhdan-Lusztig bases have positive integrality properties, see [66] and related publications. Indeed, this approach was masterfully carried out by Elias and Williamson who finally proved the Kazhdan-Lusztig basis conjecture for all Coxeter types [28].

▷ Categorifications are also studied in physics, e.g.:
  ▶ In Conformal Field Theory (CFT) researchers study fusion algebras, e.g. the Verlinde algebra. Examples of categorifications of such algebras are known, e.g. using categories connected to the representation theory of quantum groups at roots of unity [34], and contain more information than these algebras, e.g. the R-matrix and the quantum $6j$-symbols.
  ▶ The Witten genus of certain moduli spaces can be seen as an element of $\mathbb{Z}[[q]]$. It can be realized using elliptic cohomology, see e.g. [2].

1.2. **Higher representation theory.**

The "classical" question.

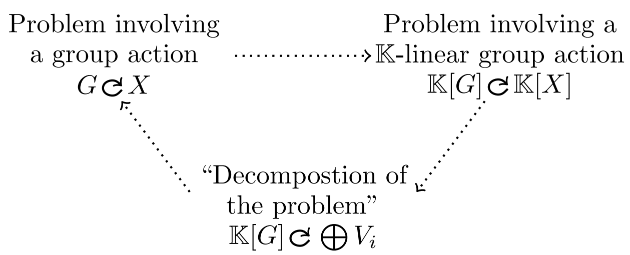
"Groups, as men, will be known by their actions." – Guillermo Moreno

The study of group actions is of fundamental importance in mathematics and related field. Sadly, it is also very hard. Luckily, there were Frobenius (∼1895 onwards) and Burnside (∼1900 onwards) and many others who gave us the following. Let $\mathbb{K}[G]$ be the group ring of a (finite) group G. Representation theory is the study of $\mathbb{K}$-linear group actions:

$$\mathrm{R}\colon \mathbb{K}[G] \longrightarrow \mathrm{End}(V), \quad \mathrm{R}(g) = \text{a "matrix" in } \mathrm{End}(V),$$

with V being some $\mathbb{K}$-vector space. We call V a G-module or a G-representation.

Representation theory approach: the analogous $\mathbb{K}$-linear problem of classifying G-modules has a satisfactory answer for many groups.



Thus, given a group G (or a ring, an algebra etc.), a classical and very interesting question is:

"Can we describe the symmetries G can act on, i.e. its representation theory?"

"Categorified" representation theory. The related, categorical question that arises is, if we can categorify the classical notions. That is:

"Can we describe the symmetries a category $\mathbf{C}$ can act on,
i.e. its representation theory?"

Moreover, the next question would be if we can categorify that again. That is:

"Can we describe the symmetries a 2-category $\mathfrak{C}$ can act on,
i.e. its 2-representation theory?"

I will give a short introduction to the basic ideas. Much more details can, for example, be found in Rouquier's paper [63]. Another also very nice introduction is the book of Mazorchuk [50].

Let A be some (group) algebra, V be an A-module and $\mathbf{V}$ be a (suitable) category. Let $2\mathbf{End}(\mathbf{V})$ denotes its associated 2-category of endofunctors. Then:

$$\text{“Classical”} \rightsquigarrow \text{“Higher”}$$

$$a \mapsto \mathrm{R}(a) \in \mathrm{End}(V) \rightsquigarrow a \mapsto \boldsymbol{\mathcal{R}}(a) \in 2\mathbf{End}(\mathbf{V})$$

$$(\mathrm{R}(a_1) \cdot \mathrm{R}(a_2))(v) = \mathrm{R}(a_1a_2)(v) \rightsquigarrow (\boldsymbol{\mathcal{R}}(a_1) \circ \boldsymbol{\mathcal{R}}(a_2))\binom{X}{\alpha} \cong \boldsymbol{\mathcal{R}}(a_1a_2)\binom{X}{\alpha}$$

A (weak) categorification of the A-module V should be thought of a categorical action of A on $\mathbf{V}$ with an isomorphism ψ such that

$$\begin{array}{ccc} [\mathbf{V}]_{\mathbb{K}} & \xrightarrow{[\boldsymbol{\mathcal{R}}_a]} & [\mathbf{V}]_{\mathbb{K}} \\ \psi \downarrow & \circlearrowleft & \downarrow \psi \\ V & \xrightarrow[\mathrm{R}(a)]{} & V \end{array}$$

commutes. Note that such a categorification again “knows” more, i.e. it “knows the relations” between the acting matrices instead of just the acting matrices. But the higher structure is not fixed in such a categorification.

But there is in a lot of cases also a “honest higher layer”. Instead of having the algebra A acting on the $\mathbb{K}$-vector space V, we want it to act on the $\mathbb{K}$-linear category $\mathbf{V}$. But in the usual spirit of categorification this picture should “upgrade”:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\text{full-grown 2-action}} & 2\mathbf{End}(\mathbf{V}) \\ [\cdot]_{\mathbb{K}} \downarrow & \nearrow_{\text{categorical action}} & \downarrow [\cdot]_{\mathbb{K}} \\ \mathrm{A} & \xrightarrow[\text{classical action}]{} & \mathrm{End}(V) \end{array}$$

Hereby we view the “uncategorified” action as a functor from A to $\mathrm{End}(V)$ (both seen e.g. as categories with one object). Then there should be a categorification $\mathfrak{A}$ of A such that we have a diagram as above - with $\mathfrak{A}$ acting via a 2-functor (which in particular fixes the higher structure).

This, sometimes called (strong) 2-action of A, works surprisingly often.

Examples of “categorified representations”. The following list is again biased. As before, the two sources [36] and [50] give several other example.

- ▷ In some sense one of the most classical example of categorified representations is provided by (versions of) the BGG category $\mathcal{O}$.
  - ▶ For instance, the regular representation of the group ring $\mathbb{C}[S_n]$ of the symmetric group S_n has a categorification given by projective functors acting on a regular block of the BGG category $\mathcal{O}$ for $\mathfrak{sl}_n$, see e.g. [8] (where the authors, of course, do not use the word “categorification”).

  - ► Integral Specht modules of S_n can be categorified in a quite similar fashion, see e.g. [37].
  - ► Similarly for categorifications of the associated Hecke algebra.
  - ► Other constructions in this spirit are known, see [50] for an overview.
- ▷ Another nowadays classical categorification is connected to categorification of the tensor product of the vector representation of (quantum) $\mathfrak{sl}_2$:
  - ► Bernstein, Frenkel and Khovanov [7] categorified the n-fold tensor product of $\mathfrak{sl}_2$ by using certain induction/restriction functors coming from cohomology rings of Grassmannians and flag varieties.
  - ► Frenkel, Khovanov and Stroppel extended this to the graded setup, and also include a categorification of the highest weight summand of the n-fold tensor product, see [29].
  - ► Recently Naisse and Vaz [54] extended this to a categorification of the Verma module for $\mathfrak{sl}_2$.
- ▷ As already mentioned above:
  - ► Based on joint work with Chuang [16], Rouquier [63] (and indecently Khovanov and Lauda [35]) gave a categorification of all simple modules of $\mathfrak{g}$ for $\mathfrak{g}$ being a simple Lie algebra.
  - ► Very much in this spirit, Webster [72] proposed a categorification of tensor products of simple $\mathfrak{g}$-modules.
  - ► Categorification in this spirit are a "big industry" nowadays (and it is not possible to summarize it in a brief fashion) and its very hard to overestimate the influence of the approach of Chuang-Rouquier and Khovanov-Lauda.
- ▷ Shortly after the groundbreaking work of (Chuang and) Rouquier, Mazorchuk and Miemietz (and their coauthors) begun a systematic study of "categorifications of finite-dimensional modules of finite-dimensional algebras" (cf. [52] or [53] and the references therein):
  - ► In [53] they defined an appropriate 2-categorical analogue of the simple representations of finite-dimensional algebras.
  - ► Several examples are know, e.g. a well-studied class of examples is given by so-called cell representations of "categorified Coxeter groups", see [51].
  - ► At the moment, even in the case of Coxeter groups, not much is known, see e.g. [41] or [49].

2. Recently finished projects

2.1. **Foams and arc algebras.**

Link and tangle invariants and representation theory. In his pioneering work [33], Khovanov introduced the so-called arc algebra H_m. One of his main purposes was to extend his celebrated categorification of the Jones polynomial [32] to tangles.

In a series of papers [11], [12], [13], [14] and [10] Brundan and Stroppel studied a generalization of the arc algebra revealing that Khovanov's arc algebra has, left aside its knot theoretical origin, interesting representation theoretical, algebraic geometrical and combinatorial properties.

This series of results has led to several variations and generalizations of Khovanov's original formulation, utilized in a large body of work by several researchers (including myself), e.g. an $\mathfrak{sl}_3$-variation considered in [48], [61], [60] and [69], and an $\mathfrak{sl}_n$-variation studied in [46] and [70], all of them having relations to (cyclotomic) KL-R algebras as in [35] or [63], and link homologies in the sense of Khovanov and Rozansky [39]. There is also the $\mathfrak{gl}_{1|1}$-variation developed in [64] with relations to the Alexander polynomial, a version coming from odd Khovanov homology [55], as well as a type **D**-version introduced in [21] and [23] with connections to the representation theory of Brauer's centralizer algebras and orthosymplectic Lie superalgebras, see e.g. [22].

Thus, it is a worthwhile question to understand the arc algebra and its cousins in a better way.

Functoriality. In Khovanov's setup it makes sense to ask if cobordisms between tangles correspond to natural transformations between bimodules. Or said in other words, whether there is a 2-functor from the 2-category of tangles to a certain 2-category of $H = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} H_m$-bimodules. This is often called functoriality.

Sadly, Khovanov's original construction is not functorial, but only functorial up to signs. A solution to this problem was provided by Blanchet [9]. He formulated Khovanov's link homology using certain singular cobordisms, that we call ($\mathfrak{gl}_2$-)foams, which, by construction, include highly non-trivial signs fixing the functoriality of Khovanov's link homology. Moreover, Blanchet's formulation fits neatly into the framework of graded 2-representations of the categorified quantum group in the sense of [35], as it was shown in [45].

Indeed, the functorial properties of Khovanov homology (which, as mentioned above, are connected to smooth topology in dimension four) are quite hard to nail down explicitly. Moreover, an algebraic rather than combinatorial/topological way to fix and see the subtle signs was not know before we started working on the paper [25].

Functoriality, web and arc algebras. Using Blanchet's construction it makes sense to define "foamy" versions $\mathfrak{W}$ of Khovanov's arc algebra, which we call ($\mathfrak{gl}_2$-)web algebras in [25]. Unfortunately calculating in $\mathfrak{W}$ is very hard. Indeed, it is not even clear what a basis of $\mathfrak{W}$ is - left aside the question how to rewrite an arbitrary foam in terms of some basis. Thus, the main purpose of the joint work with Ehrig and Stroppel [25] is to give an algebraic counterpart of $\mathfrak{W}$, denoted by $\mathfrak{A}$, where these questions about bases are easy. We call the algebraic counterpart, which are built up using certain combinatorics of arc diagrams, the Blanchet-Khovanov algebra.

The main result of the paper [25] is then:

Theorem. There is an equivalence of graded, $\mathbb{K}$-linear 2-categories

$$\boldsymbol{\Phi}: \mathfrak{W}\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}} \xrightarrow{\cong} \mathfrak{A}\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}}$$

(between certain categories of bimodules of the two algebras) induced by an isomorphism of graded algebras

$$\Phi: \mathfrak{W}^{\circ} \to \mathfrak{A}.$$

(Where $\mathfrak{W}^{\circ}$ is a certain subalgebra of $\mathfrak{W}$.) □

This provides a direct link between the topological and the algebraic point of view. As a consequence, computations (which are hard to do in practice on the topological side) can be done on the algebraic side, whereas the associativity (a non-trivial fact on the algebraic side) is clear from the topological point of view.

Note also that Khovanov homology, by birth, can be interpreted to live in (complexes of bimodules in) $\mathfrak{W}\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}}$ and the theorem above gives a way to calculate (subtle) functorial properties of Khovanov homology.

Comparison of the various ways to fix functoriality. Blanchet's fix of functoriality is not unique. There are several variants which are functorial and, when specializing to the setup of links and forgetting the higher structure of link cobordisms, give the same invariant as Khovanov's original invariant. The follow-up work [24] compares all of these and reads as follows:

Let $\mathtt{P} = \{\boldsymbol{\alpha}, \boldsymbol{\tau}^{\pm 1}, \boldsymbol{\omega}_+^{\pm 1}, \boldsymbol{\omega}_-^{\pm 1}\}$ be a set of generic parameters. In [24] we introduced a $\mathtt{P}$-version of singular topological quantum field theories (TQFTs) which we use to define a 4-parameter foam 2-category $\mathfrak{F}[\mathtt{P}]$, that is a certain 2-category of topological origin. We obtain from $\mathfrak{F}[\mathtt{P}]$ several specializations. Among the specializations of this 4-parameter version one can find the main foam 2-categories studied in the context of higher link and tangle invariants:

- Khovanov/Bar-Natan's cobordisms (see [33] or [6]) can be obtained by specializing $\boldsymbol{\alpha} = 0$, $\boldsymbol{\tau} = 1$, $\boldsymbol{\omega}_+ = 1$, $\boldsymbol{\omega}_- = 1$,
- Caprau's "foams" (see [15]) by specializing $\boldsymbol{\alpha} = 0$, $\boldsymbol{\tau} = 1$, $\boldsymbol{\omega}_+ = i$, $\boldsymbol{\omega}_- = -i$,
- Clark-Morrison-Walker's disoriented cobordisms (see [17]) by specializing $\boldsymbol{\alpha} = 0$, $\boldsymbol{\tau} = 1$, $\boldsymbol{\omega}_+ = i$, $\boldsymbol{\omega}_- = -i$, and
- Blanchet's foams (see [9]) by specializing $\boldsymbol{\alpha} = 0$, $\boldsymbol{\tau} = -1$, $\boldsymbol{\omega}_+ = 1$, $\boldsymbol{\omega}_- = -1$.

We write for these theories **KBN**, **Ca**, **CMW** and **Bl** respectively.

We also study the web algebra $\mathfrak{W}[\mathtt{P}]$ corresponding to $\mathfrak{F}[\mathtt{P}]$, i.e. an algebra which has an associated 2-category of certain bimodules giving a (fully) faithful 2-representation of $\mathfrak{F}[\mathtt{P}]$. (Similarly for any specialization of $\mathtt{P}$.)

For $\mathtt{Q} = \{\boldsymbol{\alpha}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}^{\pm 1}\}$ (obtained by specializing $\mathtt{P}$ via $\boldsymbol{\tau} = \boldsymbol{\varepsilon}\boldsymbol{\omega}^2$, $\boldsymbol{\omega}_+ = \boldsymbol{\omega}$ and $\boldsymbol{\omega}_- = \boldsymbol{\varepsilon}\boldsymbol{\omega}$ with $\boldsymbol{\varepsilon} = \pm 1$) we define an algebraic model simultaneously for $\mathfrak{F}[\mathtt{Q}]$ and $\mathfrak{W}[\mathtt{Q}]$, that is, an arc algebra $\mathfrak{A}$ encoding algebraically/combinatorially the topological data coming from $\mathfrak{F}[\mathtt{Q}]$ and $\mathfrak{W}[\mathtt{Q}]$. The foam 2-category, web and arc algebra, still contain our four main examples as specializations. We call the $\boldsymbol{\varepsilon} = 1$ specializations the $\mathfrak{sl}_2$ specializations and the $\boldsymbol{\varepsilon} = -1$ specializations the $\mathfrak{gl}_2$ specializations, since they correspond to the web algebras describing the tensor categories of finite-dimensional representations of the respective complex Lie algebra.

Our main result is, surprisingly, that any two specializations of $\boldsymbol{\varepsilon}, \boldsymbol{\omega}^{\pm 1}$ are isomorphic/equivalent. More precisely, if we denote by $*$ any such specialization, then (we only extend scalars to get an isomorphism of Q-algebras):

Theorem. Let $Q = \mathbb{Z}[\boldsymbol{\alpha}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}^{\pm 1}], \boldsymbol{\varepsilon} = \pm 1$. There are graded algebra isomorphisms

$$\Psi\colon \mathfrak{A}[\mathtt{Q}] \xrightarrow{\cong} \mathfrak{A}[*] = \mathfrak{A}_R[*] \otimes_{\mathbb{Z}} Q.$$

(Similarly for the corresponding web algebras.) □

(Additionally one can also specialize $\boldsymbol{\alpha}$.) From this we obtain:

Theorem. The isomorphisms from above induce isomorphisms of graded, Q-linear 2-categories (of certain graded bimodules)

$$\boldsymbol{\Psi}\colon \mathfrak{A}[\mathtt{Q}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}} \xrightarrow{\cong} \mathfrak{A}[*]\text{-}\mathbf{biMod}^p_{\mathrm{gr}},$$

giving on the topological side equivalences of graded, Q-linear 2-categories

$$\mathfrak{A}[\mathtt{Q}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}} \cong \mathfrak{F}[\mathtt{Q}] \cong \mathfrak{F}[*] \cong \mathfrak{A}[*]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}.$$

(Similarly for any further simultaneous specialization of $\boldsymbol{\alpha}$.) □

An almost direct consequence of the above results is:

Corollary. As special cases: the **KBN**, **Ca**, **CMW** and **Bl** setups are all equivalent (when one works over the ground ring $\mathbb{Z}[i]$). ■

As an application of our explicit isomorphisms/equivalences we discuss how one can obtain a "singular TQFT model" for the graded BGG parabolic category $\mathcal{O}$ for a certain two-block parabolic in type **A** (this is the category used in the Lie theoretical construction of Khovanov homology, see e.g. [67] or [68]). Another application is that the higher tangle invariants constructed from the various 2-categories are the same (they get identified by the above equivalence) and not just the associated link homologies. Moreover, the $\mathfrak{gl}_2$ specializations of these tend to be functorial with respect to link cobordisms, as e.g. the **Ca**, **CMW** and **Bl** specializations, while the $\mathfrak{sl}_2$ versions are usually not, as e.g. the **KBN** specialization (see e.g. [31]). Using our explicit translation between these, we give a way to make the Khovanov complex associated to links (via the famous cube construction) functorial without changing its simple framework (by changing the bimodule structure).

2.2. Categorical representations of dihedral groups.

Motivation: "higher representation theory". An essential problem in classical representation theory is the classification of the simple representations of any given algebra, i.e. the parametrization of their isomorphism classes and the explicit construction of a representative of each class.

In 2-representation theory, the actions of algebras on vector spaces are replaced by functorial actions of 2-categories on certain additive or abelian 2-categories. (See also Subsection 1.2 above.)

Examples are 2-representations of the 2-categories which categorify representations of quantum groups, due to (Chuang-)Rouquier [63] and Khovanov-Lauda [35], and 2-representations of the 2-category of Soergel bimodules, which categorify representations of Hecke algebras of Coxeter groups.

Mazorchuk and Miemietz [53] defined an appropriate 2-categorical analogue of the simple representations of finite-dimensional algebras, which they called simple transitive 2-representations (of finitary 2-categories).

Recall that representation theory of finite groups had a major impact on modern mathematics and related fields. The group ring of a finite group is a blueprint example of a finite-dimensional algebra, and, following Mazorchuk and Miemietz, it makes sense to ask whether we can understand its categorical representation theory.

This problem is completely out of reach at the moment. And even if one focuses on Coxeter groups (which are a well-behaved family of finite groups), their associated Hecke algebras and categorifications given by Soergel bimodules, the problem of the classification of their simple transitive 2-representations is very hard in general and not well understood, except in type A.

Thus, it is worthwhile to try to understand several instances of this in more details.

The case of small quotients. The authors of [41] studied the so-called small quotient of Soergel bimodules (this basically means that one "kills higher cells") and their simple transitive 2-representations, for all finite Coxeter types. These 2-representations are given by categories on which the bimodules act by endofunctors and the bimodule maps by natural transformations. Each of these categories is equivalent to the (projective or abelian) module category over the path algebra of a finite quiver, which can be obtained by doubling a Dynkin diagram whose type depends on the 2-representation. An almost complete classification was given in [41]:

In every finite Coxeter type of rank strictly greater than two, all the simple transitive 2-representations are equivalent to Mazorchuk and Miemietz's categorification of the cell representations of Hecke algebras [51], the so-called cell 2-representations.

Attacking rank two. The rank two case is more delicate. In dihedral type $\mathrm{I}_2(n)$, for any odd $n \in \mathbb{Z}_{>1}$, there are two, inequivalent, cell 2-representations. For such n, these exhaust all simple transitive 2-representations, up to equivalence.

When $n = 2, 4$, it was already known that the same holds, see [75]. However, when n is even and greater than four, it was shown in [41] that there exist additional simple transitive 2-representations which are not equivalent to cell 2-representations. If one has $n \notin \{12, 18, 30\}$, then there exist exactly two of these, and all simple transitive 2-representations are either one of these or cell 2-representations.

These simple transitive 2-representations were constructed intrinsically in [41]: The cell 2-representations, due to Mazorchuk and Miemietz [51], can be constructed as subquotients of the 2-category of Soergel bimodules. The additional simple transitive 2-representations of $\mathrm{I}_2(n)$, for $n > 5$ even, were constructed in [41] using an involution on the cell 2-representations, mimicking a construction in [47].

In some sense, a more explicit construction was needed to fill in the missing cases $n \in \{12, 18, 30\}$ and which hopefully works even more general. (Left aside the question whether such an explicit construction might be useful for future "applications".)

A new approach using diagrammatics. In [49], we construct all simple transitive 2-representations of the small quotient of the Soergel bimodules of type $\mathrm{I}_2(n)$ by different means: we use Elias' [26] diagrammatic version of the latter 2-category, the so-called two-color Soergel calculus. More precisely, given a Dynkin diagram of type A, D or E with a bipartition, we define two self-adjoint endofunctors $\mathbf{\Theta}_s$ and $\mathbf{\Theta}_t$ (one for each generator of $\mathrm{I}_2(n)$) on the module category over the corresponding quiver, and a natural transformation between composites of them for each generating diagram in the two-color Soergel calculus, such that all diagrammatic relations are preserved.

This construction also gives graded versions of these and graded simple transitive 2-representations over the Soergel bimodules in type $\mathrm{I}_2(\infty)$ (there exists no small quotient in this case) for any finite bipartite graph.

Moreover, in finite and infinite type, we show that two graded simple transitive 2-representations are equivalent if and only if the corresponding bipartite graphs are isomorphic (as bipartite graphs). Finally, we also determine when graded simple transitive 2-representations decategorify to isomorphic representations of the corresponding Hecke algebra, using a purely graph-theoretic property.

An interesting consequence of these results is that there are two inequivalent graded simple transitive 2-representations of type E_6 (or E_8) for the Soergel bimodules of type $\mathrm{I}_2(12)$ (or $\mathrm{I}_2(30)$), which decategorify to isomorphic representations of the associated Hecke algebra. (The two simple transitive 2-representations of type E_7 for the type $\mathrm{I}_2(18)$ Soergel bimodules have non-isomorphic decategorifications.) There are also inequivalent graded simple transitive 2-representations with isomorphic decategorifications over the type $\mathrm{I}_2(\infty)$ Soergel bimodules. To the best of our knowledge these are the first examples of such 2-representations of the same 2-category which decategorify to isomorphic representations.

3. Ongoing projects

3.1. **Foams and arc algebras outside of type** A. The approach taken in [25] and [24] to construct foams and web algebras is quite general. Thus, one might hope to see several other interesting algebras (not just arc and KL-R algebras) having a "foamy" realization.

Two striking possible generalizations, on which I work jointly with (a subset of) Ehrig, Stroppel and Wilbert, come to mind: One can either vary the underlying type (in the case discussed in Subsection 2.1 we had a $\mathfrak{gl}_2$ situation), or one can vary the underlying representations (in the case discussed in Subsection 2.1 we had a $\mathfrak{gl}_2$ situation with its defining representation).

Regarding the first point, two possible extensions to other types seem reachable. Either trying to find web algebras that categorify web spaces of type B_2 or G_2 in the sense of Kuperberg [44], or trying to find a "foamy" version of the type D arc algebra studied in the context of category $\mathcal{O}$ for $\mathfrak{so}_{2n}$ (as mentioned above in Subsection 2.1). We expect both to fit into a generalization of our construction.

Similarly, one can stay in type A, but one can consider other underlying representations than the defining or fundamental representation. For example, we expect that "symmetric foams" who categorify symmetric webs in the sense of [62] (or [71]) to fit into our setup as well.

3.2. **Categorical representations of Coxeter groups.** As stated above (in e.g. Subsection 2.2), the subject of categorical representations of Coxeter groups is widely open at the moment. Hence, there are still a lot of "gold nuggets hidden which are not very deep under the surface".

Since I believe in the future usefulness of Mazorchuk-Miemietz's approach, I want continue my research on 2-representations of Soergel bimodules and related 2-categories.

To be a bit more precise, the following are explicit questions on which I work jointly with (a subset of) Mackaay, Mazorchuk and Miemietz.

Note that the approach taken in [49] is motivated by joint work with Andersen [1]. In [1] we basically connect the Soergel bimodules in dihedral type to the representation theory of quantum $\mathfrak{sl}_2$ at roots of unity by defining a 2-representation on the category of tilting modules of quantum $\mathfrak{sl}_2$. Thus, we expect a direct connection of our work to the work of Kirillov and Ostrik [42] on "finite subgroups" of quantum $\mathfrak{sl}_2$ at roots of unity and the quantum analogue of the McKay correspondence (with potential connection to modular invariants in conformal field theory). Morally: "finite subgroups" of quantum $\mathfrak{sl}_2$ at roots of unity are "fusion subquotients" of the category of tilting modules of quantum $\mathfrak{sl}_2$, and a generalization of the connection from [1] should work.

Moreover, since the ideas from [1] heavily draw from a construction due to Khovanov and Seidel [40] (which is related to categorical actions of braid groups), one might expect connections to braid groups as well.

Another path we are exploring at the moment: We hope that the construction presented in [49] generalizes to other (infinite) Coxeter types, and, using more general graphs, we expect a similar construction to give 2-representations over Soergel bimodules of finite Coxeter type with higher order apex (i.e. which do not descend to the small quotient). We also think that our approach could extend to the construction of 2-representations of singular Soergel bimodules (as defined and studied in [73]), which are still poorly understood.

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