What is...algebra?

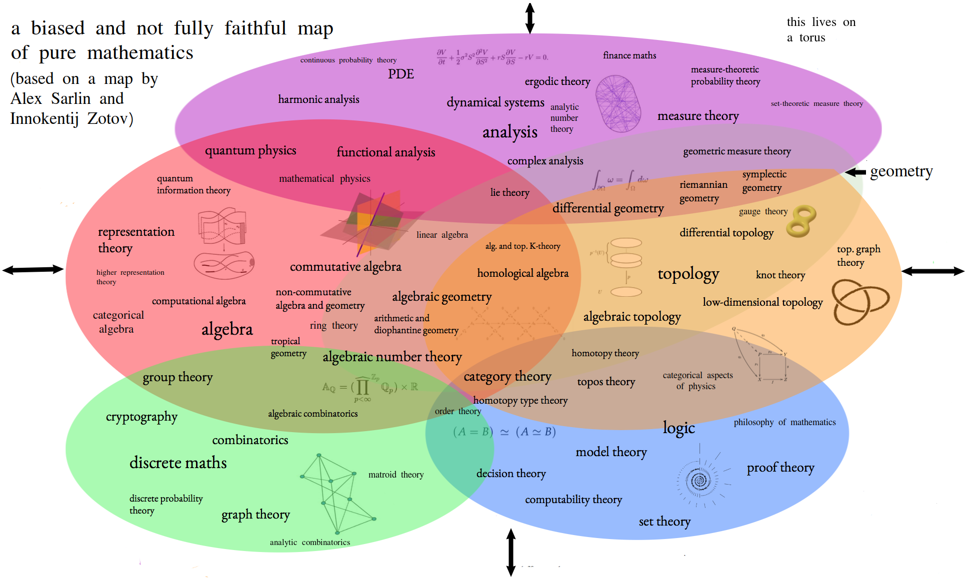
Or: How to not solve polynomial equations

The art of studying structures

a biased and not fully faithful map of pure mathematics

(based on a map by Alex Sarlin and Innokentij Zotov)

this lives on a torus



Algebra searches for the common ground of various **structures**

The evolution of algebra

(a) What are solution to polynomial equations?

$$X^2 + c_1X + c_0 = 0 \Leftrightarrow \big(c_1 = -(s_1 + s_2)\,\&\, c_0 = s_1s_2\big)$$

(b) Does a polynomial equation have a solution?

Integer solutions of $X^4 + Y^4 = Z^4$?

(c) What can be said about the nature of the solutions?

The symmetric group permutes the solutions of $X^2 + c_1X + c_0 = 0$

(d) What is the right language to study the solutions and related problems?

Addtion table of $\mathbb{F}_2$:

+	0	1
0	0	1
1	1	0

(e) What are the patterns in the language itself?

The keywords – what (a classical course in) algebra studies

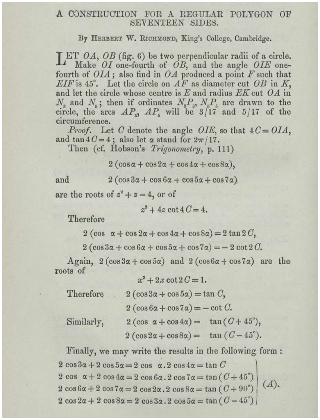
- Groups a.k.a. symmetries
  - Isomorphism theorems
  - Sylow theory
  - Permutation groups
  - ...
- Rings, fields and modules
  - Ideals
  - Prime factorization
  - Classification of abelian groups
  - ...
- Galois theory
  - Field extensions
  - Galois extensions
  - Insolvability of the quintic
  - ...

Application one – straightedge and compass constructions

Question: Find the regular *n*-gons constructible by straightedge–compass
Problem: Very hard to do explicitly
Gauss: Solve a structural problem instead

(a) There is a nice solution in terms of primes of the form $2^{2^k} - 1$

(b) The explicit construction needs a lot of work, see *e.g.* Richmond ∼1893:



A CONSTRUCTION FOR A REGULAR POLYGON OF SEVENTEEN SIDES.

By HERBERT W. RICHMOND, King's College, Cambridge.

LET OA, OB (fig. 6) be two perpendicular radii of a circle. Make OI one-fourth of OB, and the angle OIE one-fourth of OIA; also find in OA produced a point F such that EIF is 45°. Let the circle on AF as diameter cut OB in K, and let the circle whose centre is E and radius EK cut OA in N_3 and N_5; then if ordinates N_3P_3, N_5P_5 are drawn to the circle, the arcs AP_3, AP_5 will be $3/17$ and $5/17$ of the circumference.

Proof. Let C denote the angle OIE, so that $4C = OIA$, and $\tan 4C = 4$; also let α stand for $2\pi/17$.

Then (cf. Hobson's *Trigonometry*, p. 111)

$$2(\cos\alpha + \cos 2\alpha + \cos 4\alpha + \cos 8\alpha),$$

and

$$2(\cos 3\alpha + \cos 6\alpha + \cos 5\alpha + \cos 7\alpha)$$

are the roots of $z^2 + z = 4$, or of

$$z^2 + 4z \cot 4C = 4.$$

Therefore

$$2(\cos\alpha + \cos 2\alpha + \cos 4\alpha + \cos 8\alpha) = 2\tan 2C,$$
$$2(\cos 3\alpha + \cos 6\alpha + \cos 5\alpha + \cos 7\alpha) = -2\cot 2C.$$

Again, $2(\cos 3\alpha + \cos 5\alpha)$ and $2(\cos 6\alpha + \cos 7\alpha)$ are the roots of

$$x^2 + 2x\cot 2C = 1.$$

Therefore $2(\cos 3\alpha + \cos 5\alpha) = \tan C,$

$$2(\cos 6\alpha + \cos 7\alpha) = -\cot C.$$

Similarly, $2(\cos\alpha + \cos 4\alpha) = \tan(C + 45°),$

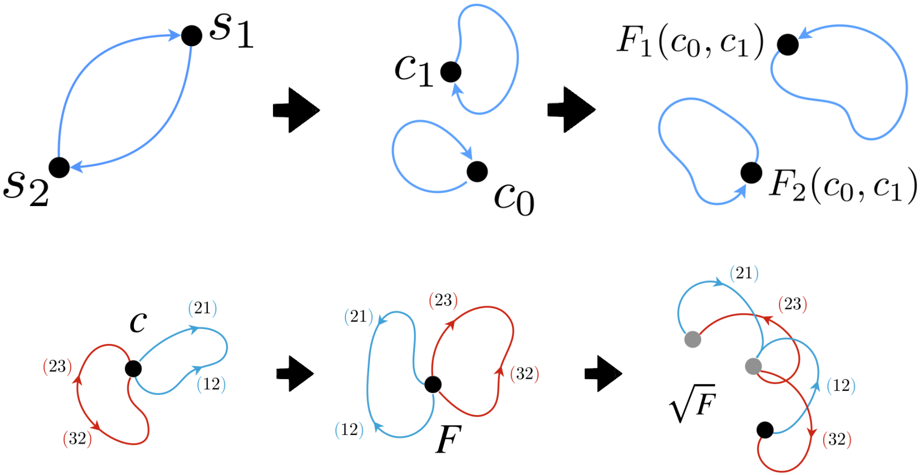
$$2(\cos 2\alpha + \cos 8\alpha) = \tan(C - 45°).$$

Finally, we may write the results in the following form:

$$\left.\begin{aligned} 2\cos 3\alpha + 2\cos 5\alpha &= 2\cos\alpha \,.\, 2\cos 4\alpha = \tan C \\ 2\cos\alpha + 2\cos 4\alpha &= 2\cos 6\alpha \,.\, 2\cos 7\alpha = \tan(C + 45°) \\ 2\cos 6\alpha + 2\cos 7\alpha &= 2\cos 2\alpha \,.\, 2\cos 8\alpha = \tan(C + 90°) \\ 2\cos 2\alpha + 2\cos 8\alpha &= 2\cos 3\alpha \,.\, 2\cos 5\alpha = \tan(C - 45°) \end{aligned}\right\} \quad (A).$$

Application two – the Abel–Ruffini theorem

There is no formula involving only $+, -, \cdot, \div, \sqrt[n]{}$ solving
$X^5 + c_4X^4 + c_3X^3 + c_2X^2 + c_1X^1 + c_0$



The proof uses the structure of symmetric groups

Thank you for your attention!

I hope that was of some help.