What is...the Jordan–Hölder theorem?

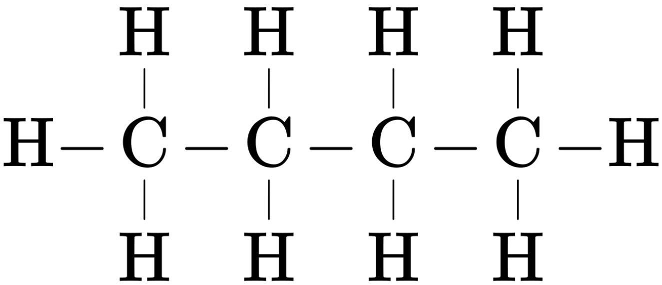
Or: The elements of group theory.

Elements of group theory

- A group G is called simple if 1 and G are the only normal subgroups
- If $1 \subsetneq N \subsetneq G$ is normal, then G/N and N are smaller groups to study, *e.g.*

$$1 \subsetneq \mathbb{Z}/6\mathbb{Z} \subsetneq \mathbb{Z}/12\mathbb{Z}, \quad \frac{\mathbb{Z}/12\mathbb{Z}}{\mathbb{Z}/6\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z} \text{ and } \mathbb{Z}/6\mathbb{Z}$$

- Simple groups are the elements of group theory



Question. Is there a group-analog of a chemical formula?

Composition series

- $N \lhd G$ means $N \subset G$ is normal
- Decomposition into smaller components N_k/N_{k+1}

$$\dots \lhd N_3 \lhd N_2 \lhd N_1 = G$$

is called a normal series, *e.g.*

$$1 \lhd \mathbb{Z}/6\mathbb{Z} \lhd \mathbb{Z}/12\mathbb{Z}, \quad \frac{\mathbb{Z}/12\mathbb{Z}}{\mathbb{Z}/6\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z} \text{ and } \frac{\mathbb{Z}/6\mathbb{Z}}{1} \cong \mathbb{Z}/6\mathbb{Z}$$

- Decomposition into the simplest components N_k/N_{k+1}

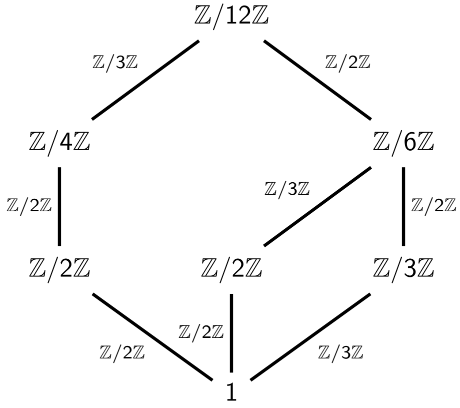
$$1 = N_n \lhd \dots \lhd N_3 \lhd N_2 \lhd N_1 = G$$

is called a composition series if N_k/N_{k+1} are simple, *e.g.*

$$1 \lhd \mathbb{Z}/2\mathbb{Z} \lhd \mathbb{Z}/6\mathbb{Z} \lhd \mathbb{Z}/12\mathbb{Z}, \frac{\mathbb{Z}/12\mathbb{Z}}{\mathbb{Z}/6\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z} \text{ and } \frac{\mathbb{Z}/6\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z}/3\mathbb{Z} \text{ and } \frac{\mathbb{Z}/2\mathbb{Z}}{1} \cong \mathbb{Z}/2\mathbb{Z}$$

Question. Do these exist? Are composition series unique?

The fundamental theorem of arithmetic



This generalizes $12 = 3 \cdot 2 \cdot 2 = 2 \cdot 3 \cdot 2 = 2 \cdot 2 \cdot 3$

For completeness: The formal statement

Let G be a finite group

(a) There is a composition series $1 = N_n \lhd \dots \lhd N_3 \lhd N_2 \lhd N_1 = G$ Existence

(b) Two composition series have the same length n and the same composition factors N_k/N_{k+1} up to reordering and isomorphism Uniqueness

(c) The composition factors N_k/N_{k+1} are invariants of G

For all G, if $1 = N_n \lhd \dots \lhd N_3 \lhd N_2 \lhd N_1 = G$ exists, then (b) and (c) still hold

That G is finite is essential for existence – *e.g.* $\mathbb{Z}$ does not have a composition series

The elements – examples of finite simple groups

- 1 is sometimes included
- $\mathbb{Z}/p\mathbb{Z}$ for p prime
- Alternating groups A_n for $n \geq 5$ (A_5 is the smallest simple non-abelian group)
- Matrix groups such as

$$\mathrm{SL}_2(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{F}_p, ad - bc = 1 \right\}$$

  (This is almost true: You have to massage them a bit to get a simple group)
- Funny exceptions, but only 26 of these (or 27, depending who you ask)

$$n \geq 5: \quad 1 \lhd A_n \lhd S_n, \quad \frac{S_n}{A_n} \cong \mathbb{Z}/2\mathbb{Z}, \quad \frac{A_n}{1} \cong A_n$$

Thank you for your attention!

I hope that was of some help.