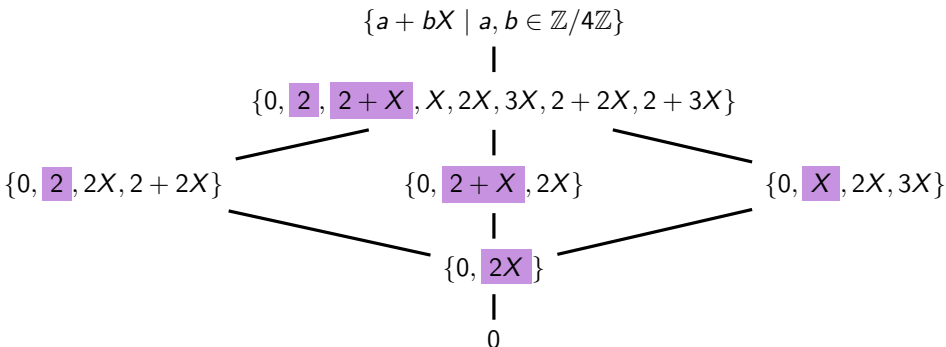


What is...the calculus of ideals?

Or: Ideals are like numbers

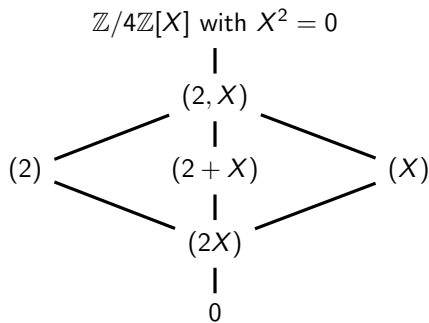
Example. $\mathbb{Z}/4\mathbb{Z}[X]$ with $X^2 = 0$



The elements outside of $\{0, 2, 2 + X, X, 2X, 3X, 2 + 2X, 2 + 3X\}$ are all invertible :

- ▶ $1 \cdot 1 = 1, 3 \cdot 3 = 9 \equiv 1 \pmod{4}$
- ▶ $(a + bX)(c + dX) \equiv 1 \pmod{4} \Leftrightarrow (a = c \in \{1, 3\} \text{ and } b = -d)$
- ▶ Example. $(3 + 2X)(3 + 2X) = 9 + 12X + 4X^2 \equiv 1 \pmod{4}$

A slick illustration

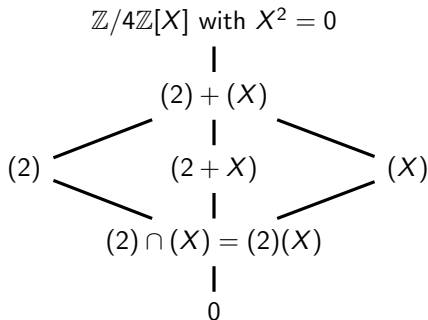


- ▶ $(2, X) \iff$ elements of the form $r \cdot 2 + r' \cdot X$
- ▶ $(2), (2 + X), (X) \iff$ elements of the form $r \cdot 2, r \cdot (2 + X), r \cdot X$
- ▶ $(2X) \iff$ elements of the form $r \cdot 2 \cdot X$

(a) $Ra = \{r \cdot a \mid r \in R\}$, $aR = \{a \cdot r \mid r \in R\}$, $RaR = \{r \cdot a \cdot r' \mid r, r' \in R\}$

(b) $(a, b, \dots) = \{r \cdot a + r' \cdot b + \dots\}$ ideal generated by a, b, \dots

Another slick illustration



- ▶ The sum $I + J$ of ideals gives an addition
 - ▶ The product IJ of ideals gives a multiplication
 - ▶ $I + J$ is a supremum, is $I \cap J$ an infimum
-

(a) $I + J = \{r \cdot i + r' \cdot j \mid r, r' \in R, i \in I, j \in J\}$, $IJ = \{r \cdot i \cdot j \mid r \in R, i \in I, j \in J\}$

(b) $I \cap J$ the set theoretic intersection **Warning. $I \cup J$ is in general not an ideal**

For completeness: The formal statement

Let R be a ring, then the collection of left ideal \mathcal{I} forms a semiring and a lattice:

- (a) \mathcal{I} has an addition $+$, \mathcal{I} has a multiplication \cdot Two operations
 - (b) $(\mathcal{I}, +)$ is an abelian monoid
 - (c) (\mathcal{I}, \cdot) is a monoid
 - (d) The two rules distribute over one another Compatibility
 - (e) (\mathcal{I}, \subset) has $+$ and \cap as a supremum respectively infimum Order
-

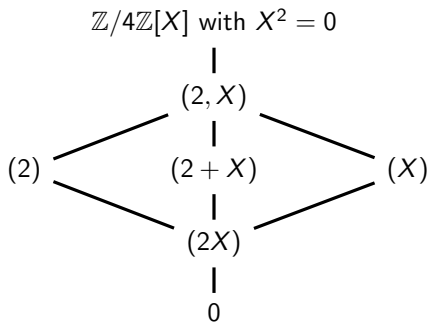
- ▶ The example of a semiring and a lattice is

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

Ideals behave like numbers!

- ▶ If R has a unit, then R is the unit of \mathcal{I}
- ▶ If R is commutative, then so is \mathcal{I}

Generalize properties of numbers



- ▶ $(2, X)$ is a maximal ideal
- ▶ $(2X)$ is a minimal ideal
- ▶ $(2, X)$ is a prime ideal
- ▶ (2) , $(2 + X)$, (X) and $(2X)$ are principal ideals
- ▶ *etc.*

Thank you for your attention!

I hope that was of some help.