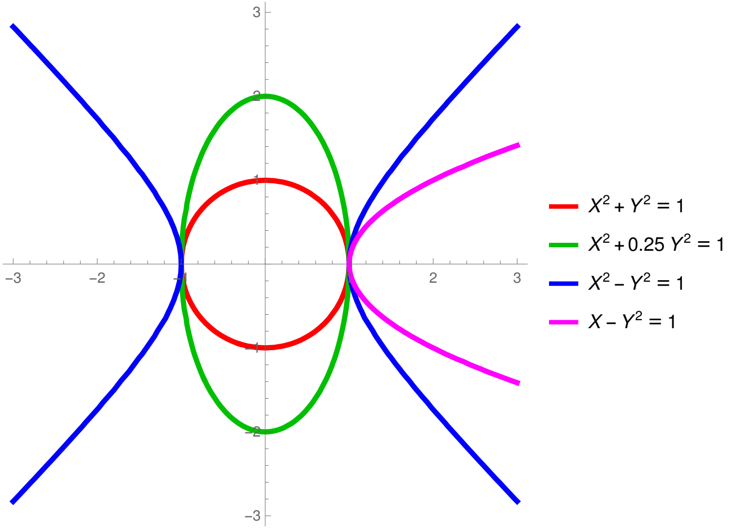
What is...Hilbert's basis theorem?

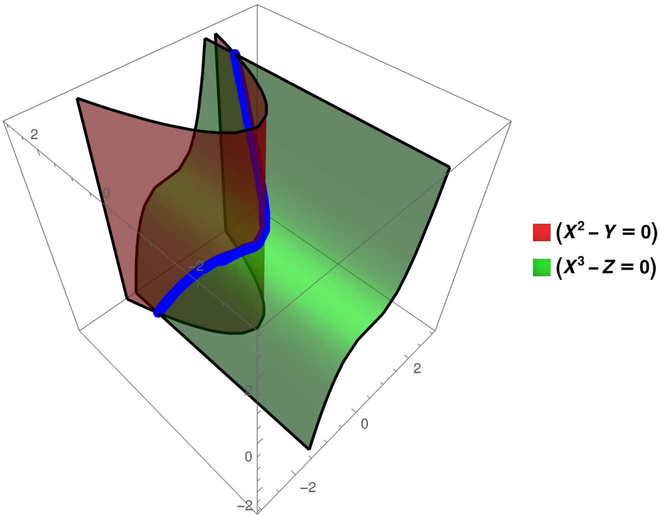
Or: Algebra meets geometry (once again)

The conic sections – circles, ellipses, hyperbola and parabola



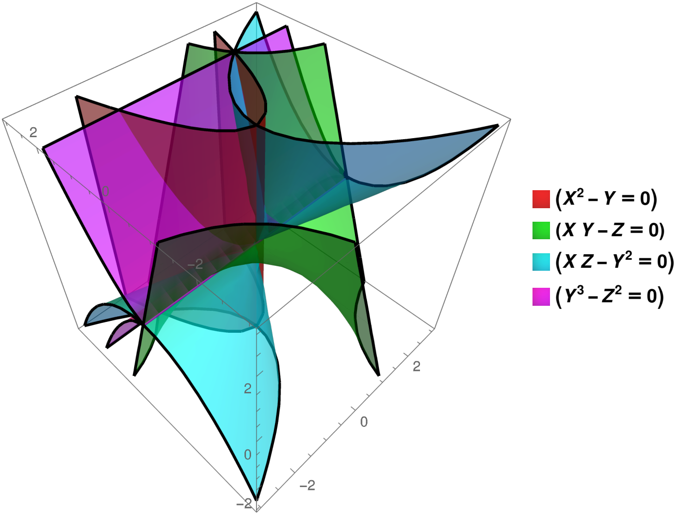
Conic sections are all algebraic varieties – solutions to polynomial equations

More than one equation



Question. Are algebraic varieties intersections of finitely many polynomial equations?

Reformulate in terms of generators of ideals



The point. $(X^2 - Y, X^3 - Z) = (X^2 - Y, XY - Z, XZ - Y^2, Y^2 - Z^3)$

For completeness: The formal statement

If R is Noetherian, then so is $R[X]$

Consequently, also $R[X_1, ..., X_n]$ is Noetherian

Hence, algebraic varieties are the common roots of finitely many polynomials

- Algebraic variety $X =$ the zero locus V of some set of polynomials
- Every X has an associated ideal $I(X)$ such that $X = V(I(X))$ From X to I
- The converse is almost true From I to X – see Hilbert's Nullstellensatz
- Noetherian = every ideal is finitely generated
- Every field $\mathbb{K}$ is Noetherian, $\mathbb{Z}$ is Noetherian, so Hilbert tells us that $\mathbb{K}[X_1, ..., X_n]$ and $\mathbb{Z}[X_1, ..., X_n]$ are Noetherian
- Rings that are not Noetherian tend to be "large", *e.g.* $\mathbb{Z}[X_1, X_2, ...]$

Hilbert's proof and the most modern proofs are non-constructive; a constructive version follows from Gröbner theory:

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GroebnerBasis[{X^2 - Y, X^3 - Z}, {Z, Y, X}]
GroebnerBasis[{X^2 - Y, X^3 - Z}, {X, Y, Z}]
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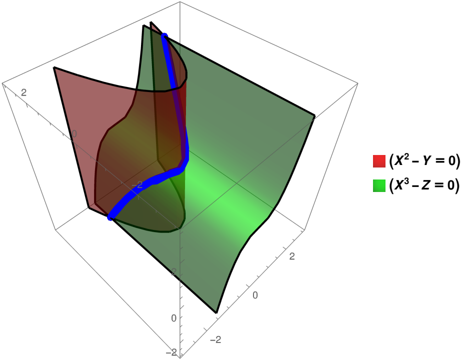
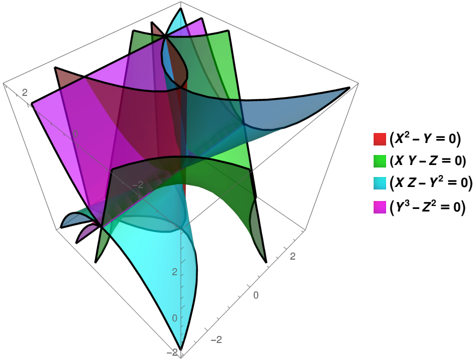
$\{-X^2 + Y, -X^3 + Z\}$

$\{Y^3 - Z^2, -Y^2 + X Z, X Y - Z, X^2 - Y\}$





Compare. Proof as a visualization exercise *vs.* a Gröbner algorithm

Thank you for your attention!

I hope that was of some help.