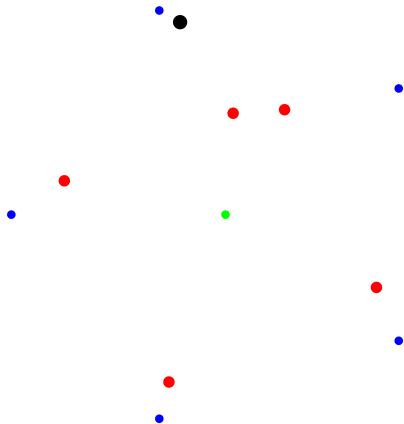


What are...normal and separable extensions?

Or: Linear factors matter

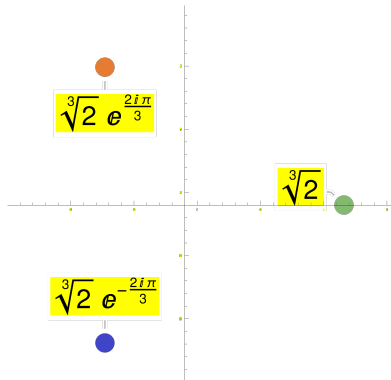
Groups actions on roots



Groups act on roots of polynomials
How is this reflected in field extensions?

An ill-behaved example

Roots of $X^3 - 2$
 $\{\sqrt[3]{2}, \zeta\sqrt[3]{2}, \zeta^2\sqrt[3]{2}\}$:
 $\zeta = e^{2\pi i/3}$



► $\mathbb{Z}/2\mathbb{Z}$ acts on the roots of $X^3 - 2$:

$$\zeta \leftrightarrow \zeta^2$$

► This information is lost in $\mathbb{Q}(\sqrt[3]{2})$

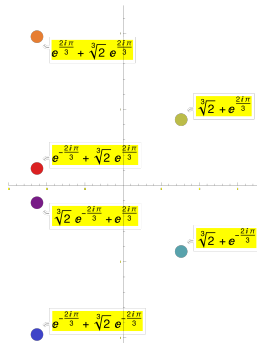
► $X^3 - 2$ does not split in $\mathbb{Q}(\sqrt[3]{2})$

A well-behaved example

Roots of $f = X^6 + 3X^5 + 6X^4 + 3X^3 + 9X + 9$

$$\{\zeta^a + \zeta^b \sqrt[3]{2} \mid a = 1, 2, b = 0, 1, 2\} \quad :$$

$$\zeta = e^{2\pi i/3}$$



- ▶ The symmetric group S_3 **permutes** the roots of f :

$$\zeta \leftrightarrow \zeta^2, \quad \begin{array}{ccc} & \zeta^{\sqrt[3]{2}} & \\ & \downarrow & \swarrow \zeta^{\sqrt[3]{2}} \\ & \zeta^2 \sqrt[3]{2} & \nearrow \zeta^{\sqrt[3]{2}} \end{array}$$

- ▶ This information is **present** in $\mathbb{Q}(\zeta + \sqrt[3]{2}) = \mathbb{Q}(\zeta, \sqrt[3]{2})$

- ▶ f **does split** in $\mathbb{Q}(\zeta + \sqrt[3]{2})$

For completeness: The formal definitions

An algebraic field extension $\mathbb{K} \subset \mathbb{L}$ is normal over \mathbb{K} if

every irreducible $f \in \mathbb{K}[X]$ with a root in \mathbb{L} splits

An algebraic field extension $\mathbb{K} \subset \mathbb{L}$ is separable over \mathbb{K} if for all $x \in \mathbb{L}$

the minimal polynomials $m_x \in \mathbb{K}[X]$ have $\partial_x m_x \neq 0$

- ▶ Minimal normal extensions exist and are unique **Existence and uniqueness**
- ▶ Every algebraic extension of a field of characteristic zero or a finite field is separable **Morally, separable is a non-condition**

No multiple roots

For $f = a_n X^n + \dots + a_0$ define $\partial_X f = n a_n X^{n-1} + \dots + a_1$

- ▶ ∂_X satisfies linearity:

$$\partial_X(a \cdot f + b \cdot g) = a \cdot \partial_X f + b \cdot \partial_X g$$

- ▶ ∂_X satisfies the product rule:

$$\partial_X(fg) = \partial_X(f)g + f\partial_X(g)$$

- ▶ For irreducible $f \in \mathbb{K}[X]$ we have

$$(\partial_X f \neq 0) \Leftrightarrow f \text{ has no multiple roots}$$

Thank you for your attention!

I hope that was of some help.