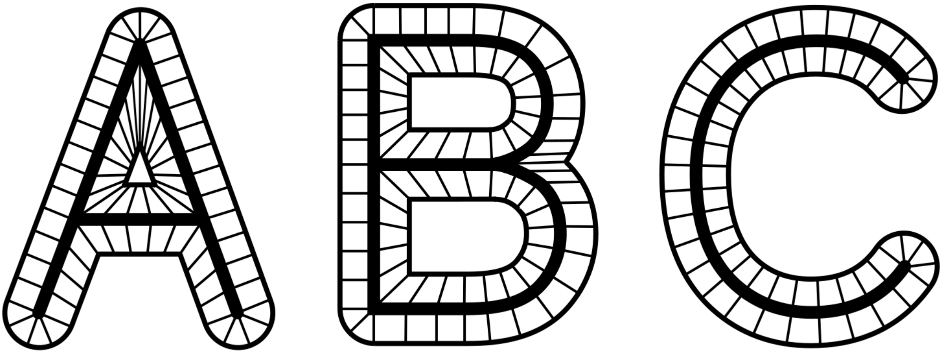
What is...homotopy?

Or: The same shape!?

The homotopy types of the Latin alphabet



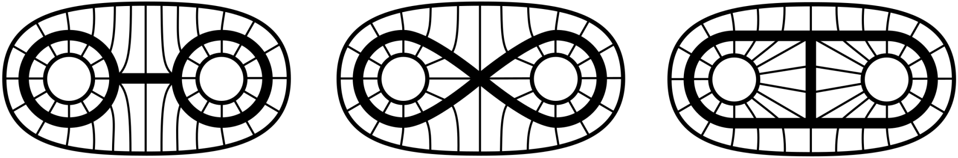
Homotopy types of the graphs underlying the alphabet:

Genus 0	Genus 1	Genus 2
CEFGHIJKLMNSTUVWXYZ	ADOPQR	B

Question. How to make this precise?

A projection in topology

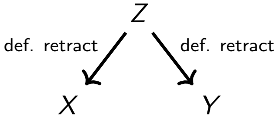
- Retraction $r\colon X \to X$ with $r^2 = r$ Idempotent
- r is a projection onto its image $A \subset X$: $r(X) = A$ and $r|_A = A$ Projection
- Deformation retraction $h_t\colon X \to X$ with $h_0 = \mathrm{id}$ and $h_1 = r$ a retraction; the family h_t is continuous, *i.e.* $X \times [0,1] \to X, (x,t) \mapsto h_t(x)$ is continuous
  In topology everything should be continuous

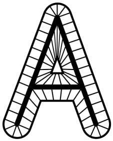
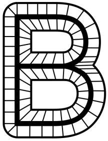
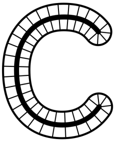


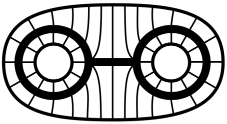
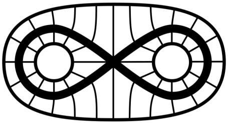
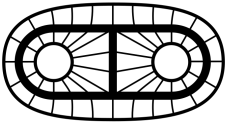
Three deformation retracts of the same space

Equivalent shapes

$X \simeq Y$ if $\exists Z$, such that



 $\simeq$ circle  $\simeq$ figure eight  $\simeq$ point

 $\simeq$  $\simeq$ 

For completeness: A formal definition

Two continues maps $f, f' \colon X \to Y$ are homotopic $f \simeq f'$ if:

(a) There exists a continuous $h_t \colon X \to Y$ with $h_0 = f$ and $h_1 = f'$

(b) $X \times [0,1] \to X, (x,t) \mapsto h_t(x)$ is continuous

Two topological spaces X, Y are homotopy equivalent $X \simeq Y$ if:

(a) There exists continuous $f \colon X \to Y$ and $g \colon Y \to X$

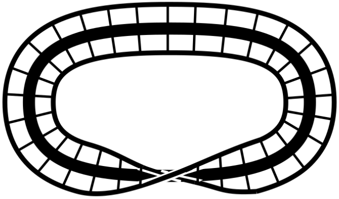
(b) $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$

$X \simeq Y$ if and only if both are homeomorphic to deformation retracts of a space Z

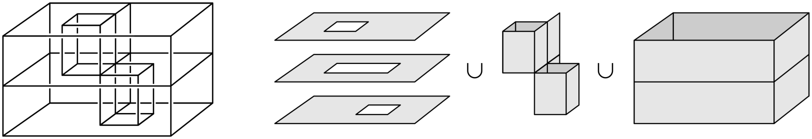
The correct notion for algebraic topology:

- Homology and cohomology (singular) are invariant under homotopy
- The fundamental group and homotopy groups are invariant under homotopy (for reasonable spaces)

Careful with “equal”



$\simeq$ circle



$\simeq$ point

In topology the is no “obviously correct” version of equal

Homeomorphic $\Rightarrow$ homotopic, but not *vice versa* in general

Thank you for your attention!

I hope that was of some help.