What is...homology categorifying?

Or: Modules, polynomials and numbers

Homology and Hilbert–Poincaré and Euler

An honest module	homology H_*
$\downarrow$	$\downarrow$
Graded dimensions	Hilbert–Poincaré polynomial P
$\downarrow$	$\downarrow$
$t=-1$	Euler χ

- H_* is a graded $\mathbb{Z}$-module An honest module

$$H_*(\text{Klein bottle } K) = H_0(K) \oplus H_1(K) \oplus H_2(K) \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus 0$$

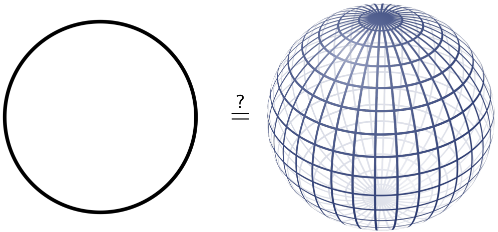
- P is a polynomial in $\mathbb{N}[t]$ obtained by taking dimension when working with $\mathbb{Q}$

$$P(K)(t) = \dim_{\mathbb{Q}} \big(H_*(K) \otimes_{\mathbb{Z}} \mathbb{Q}\big) = 1 + t$$

- χ is a number obtained by $t=-1$

$$\chi(K) = P(K)(-1) = 0$$

Homology vs. Hilbert–Poincaré vs. Euler – part 1



- H_* distinguishes spheres

$$H_n(S^d) \cong \begin{cases} \mathbb{Z} & n = 0, d \\ 0 & \text{else} \end{cases}$$

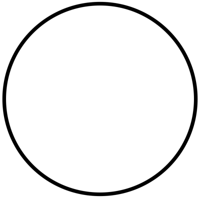
- P distinguishes spheres

$$P(S^d) = 1 + t^d$$

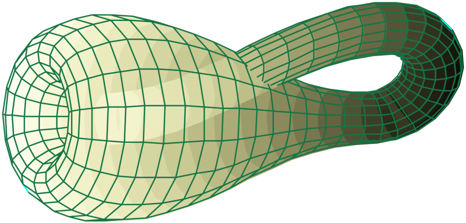
- χ does not distinguish spheres:

$$\chi(S^d) = \begin{cases} 2 & d \text{ even} \\ 0 & d \text{ odd} \end{cases}$$

Homology vs. Hilbert–Poincaré vs. Euler – part 2



$\stackrel{?}{=}$



- H_* distinguishes S^1 from K

$$H_*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_*(K) \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$$

- P does not distinguish S^1 from K

$$P(S^1) = P(K) = 1 + t$$

- χ does not distinguish S^1 from K

$$\chi(S^1) = \chi(K) = 0$$

For completeness: A formal definition/statement

- Singular homology $H_*(X)$ is a graded $\mathbb{Z}$-module homotopy invariant

$$H_*(X) = \bigoplus_{i \in \mathbb{N}} H_i(X)$$

- If $\dim_{\mathbb{Q}}(H_i(X) \otimes_{\mathbb{Z}} \mathbb{Q})$ is finite $\forall i$, then we get a homotopy invariant

$$P(X)(t) = \sum_{i \in \mathbb{N}} \dim_{\mathbb{Q}}(H_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}) t^i$$

In general this is a formal power series, not a polynomial

- For X with finite $P(X)(t)$ we get a homotopy invariant

$$\chi(X) = P(X)(-1)$$

For X being a finite cell complex this agrees with the "alternating-sum-of-cells" definition of χ

Homology vs. Hilbert–Poincaré vs. Euler – part 3

- H_* is a functor It knows maps as well
- From this one gets the Lefschetz numbers $\Lambda(f)$ for $f\colon X \to X$
- For a reasonable space X: $\Lambda(f) = 0$ if and only if f has a fixed point
- There is nothing comparable for P or χ

Using this one can give a short proof of the Brouwer fixed point theorem since

$$\Lambda(f\colon D^n \to D^n) \neq 0$$

Thank you for your attention!

I hope that was of some help.