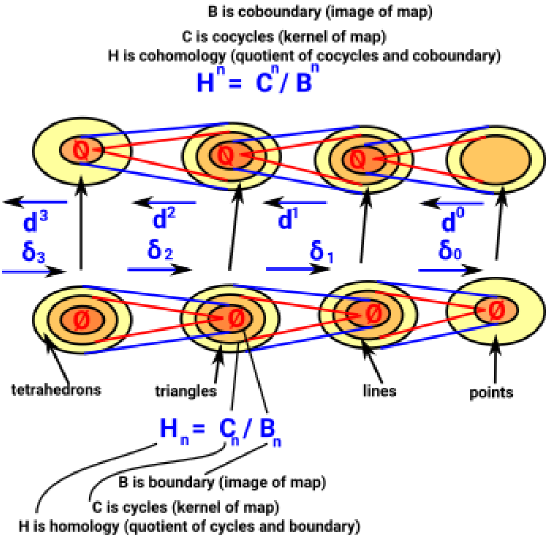
What is...cohomology?

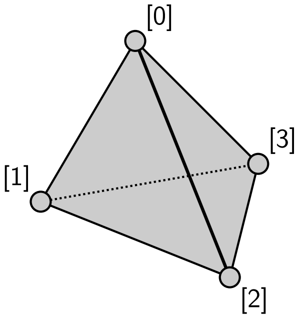
Or: Reversing arrows

Homology goes down, cohomology goes up



Why does homology H_* prefer a direction? For no good reason

Homology goes down



$$[0,1,2,3] \mapsto [0,1,2]+[0,1,3]+[0,2,3]+[1,2,3]$$

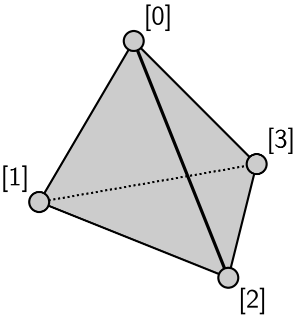
$$[0,1,2] \mapsto [0,1]+[0,2]+[1,2]$$

$$[0,1] \mapsto [0]+[1]$$

- tetrahedron $\xrightarrow{\text{homology}}$ sum of triangles $3 \to 2$
- triangle $\xrightarrow{\text{homology}}$ sum of lines $2 \to 1$
- line $\xrightarrow{\text{homology}}$ sum of points $1 \to 0$

(Here with $\mathbb{Z}/2\mathbb{Z}$ coefficients so no need to worry about orientations.)

Cohomology goes up



$[0,1,2] \mapsto [0,1,2,3]$

$[0,1] \mapsto [0,1,2] + [0,1,3]$

$[0] \mapsto [0,1] + [0,2] + [0,3]$

- triangle $\xrightarrow{\text{cohomology}}$ sum of tetrahedrons $2 \to 3$
- line $\xrightarrow{\text{cohomology}}$ sum of triangles $1 \to 2$
- point $\xrightarrow{\text{cohomology}}$ sum of lines $0 \to 1$

(Here with $\mathbb{Z}/2\mathbb{Z}$ coefficients so no need to worry about orientations.)

For completeness: A formal definition

Let X be any topological space

- The nth singular co chain group is

$$C^n = C^n(X) = \mathbb{Z}\{\text{singular } n\text{-cosimplices}\} = \hom\big(\mathbb{Z}\{\sigma_n \colon \Delta^n \to X\}, \mathbb{Z}\big)$$

- The nth singular co chain map is

$$\partial^n \colon C^n \to C^{n-1}, \quad \partial^n = (\partial_n)^*$$

- The ith singular co homology is

$$H^n = H^n(X) = \ker(\partial^n)/\mathrm{im}(\partial^{n-1})$$ Homology has $\mathrm{im}(\partial_{n+1})$

- Singular cohomology is a homotopy/homeomorphism invariant

Simplicial and cellular cohomology also exists

Singular cohomology=simplicial cohomology=cellular cohomology for any reasonable X

Singular cohomology is general, simplicial cohomology is computable for machines, cellular is computable for humans

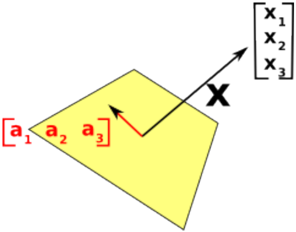
Linear forms instead of vectors

- Note that

$$C^n = \hom(C_n, \mathbb{Z}), \quad \partial^n = (\partial_n)^*$$

  This reverses all the arrows

- This is the same idea of defining dual vectors as linear forms



  Transpose vectors

- This approach prefers homology over cohomology

Thank you for your attention!

I hope that was of some help.