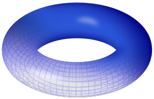
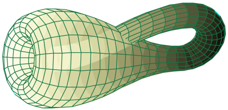
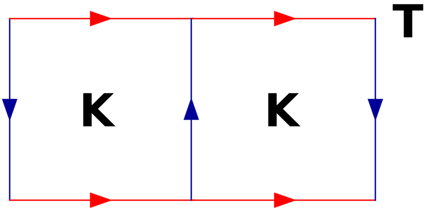
What is...a covering space?

Or: A topological Galois correspondence

Algebra should reflect topology

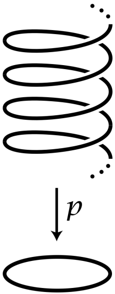
The torus T and the Klein bottle K:

$\mathbb{Z}^2 \cong \pi_1 \Bigg($  $\Bigg) \subset_{\text{subgroup}} \pi_1 \Bigg($  $\Bigg) \cong \langle a, b \mid abab^{-1} \rangle$



Question What topology corresponds to subgroups of $\pi_1(X)$?

Helixes and circles

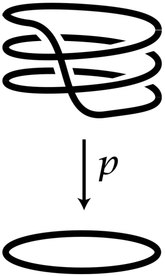


The helix $\mathbb{R}$ covers the circle S^1:

$$p_\infty \colon \mathbb{R} \to S^1, t \mapsto \exp(2\pi i t)$$

- Neighborhoods in $\mathbb{R}$ are mapped nicely to neighborhoods in S^1
  Locally the same
- Each $z \in S^1$ has $\mathbb{Z}$-sheets $p^{-1}(z) \xleftrightarrow{\text{“1:1”}} \mathbb{Z}$ Unwrapping
- This plays a crucial role in the calculation of $\pi_1(S^1) \cong \mathbb{Z}$ Relation to π_1

Winding around the circle S^1



The circle S^1 covers the circle S^1:

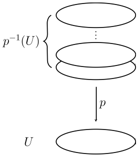
$$p_n\colon S^1 \to S^1, z \mapsto z^n$$

- Neighborhoods in S^1 are mapped nicely to neighborhoods in S^1 Locally the same
- Each $z \in S^1$ has $n\mathbb{Z}$-sheets $p^{-1}(z) \xleftrightarrow{\text{“1:1”}} n\mathbb{Z}$ Unwrapping
- $n\mathbb{Z}$ are precisely the non-trivial subgroups of $\mathbb{Z} \cong \pi_1(S^1)$ Relation to π_1

For completeness: A formal definition/statement

A covering of a topological space X is a pair $(\tilde{X}, p)$ of a topological space $\tilde{X}$ and a continues surjection $p\colon \tilde{X} \to X$ such that they are locally the same: each point $x \in X$ has an open neighborhood U with $p^{-1}(U)$ being a union of disjoint open sets (sheets), each of which is mapped homeomorphically onto U

Sheets:



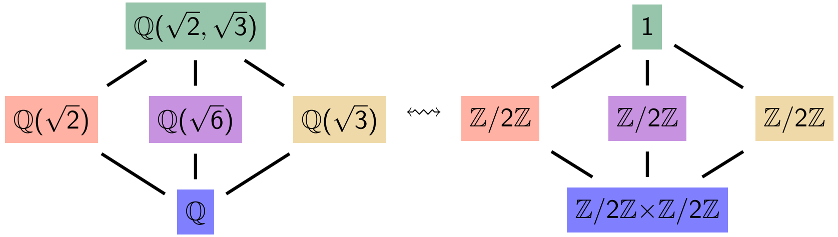
- Galois correspondence For X reasonable there is a bijection

$$\left\{\begin{matrix}\text{path-connected}\\ \text{coverings}\end{matrix}\right\}/\text{iso.} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{subgroups of}\\ \pi_1(X)\end{matrix}\right\}/\text{conj.}$$

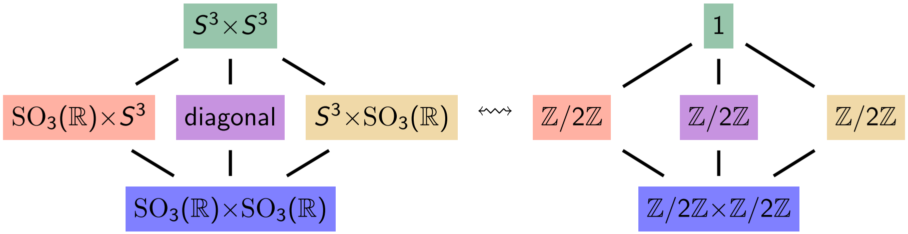
- The trivial cover of X is X (the "smallest cover") and corresponds to $\pi_1(X)$
- The universal cover of X (the "biggest cover") and corresponds to 1

Galois in topology

Field extensions and subgroups of the Galois group, *e.g.*



Covers and subgroups of the fundamental group, *e.g.*



Thank you for your attention!

I hope that was of some help.