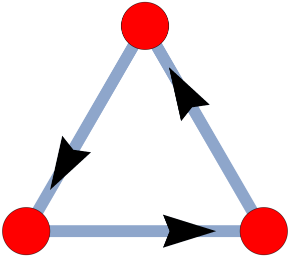
What is...a Cayley complex?

Or: Groups geometrically

Cayley graphs

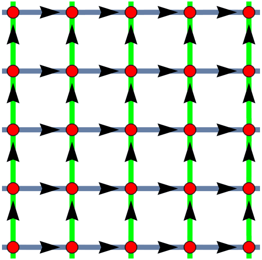
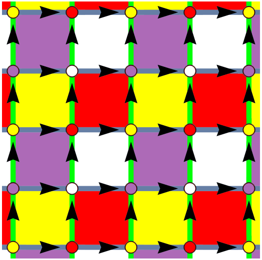
$\mathbb{Z}/3\mathbb{Z} \cong \langle a \mid a^3 = 1 \rangle \leftrightsquigarrow \Gamma_{\mathbb{Z}/3\mathbb{Z}} =$  $\pi_1(\Gamma_{\mathbb{Z}/3\mathbb{Z}}) \cong \mathbb{Z}$

The Cayley graph Γ_G of $G = \langle S \mid R \rangle$ (S generators, R relations) is the CW complex with:

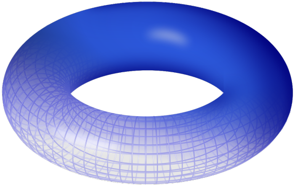
- 0 cells being the elements of G
- 1 cells being edges from g to gs for $s \in S$

Observations X_G does not take R into account, and $\pi_1(X_G)$ is a free group

Discs and relations for $\mathbb{Z}\times\mathbb{Z} \cong \langle a, b \mid aba^{-1}b^{-1} = 1\rangle$

$\Gamma_{\mathbb{Z}\times\mathbb{Z}} =$  $\longrightarrow \mathbb{Z}\times\mathbb{Z} \circlearrowright \tilde{X}_{\mathbb{Z}\times\mathbb{Z}} \simeq$ 

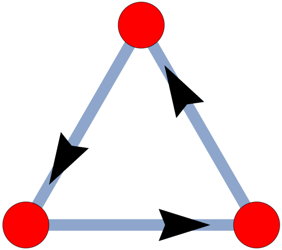
- $\tilde{X}_G$ is constructed by gluing discs for each $g \in G$ and $r \in R$ Discs
- G acts on $\tilde{X}_G$, so we obtain $X_G = \tilde{X}_G/G$ Relations

$X_{\mathbb{Z}\times\mathbb{Z}} \simeq S^1 \times S^1 \simeq$ 

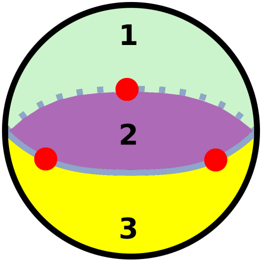
- Covering $\tilde{X}_G \to X_G$ and $\pi_1(\tilde{X}_G) \cong 1$ gives $\pi_1(X_G) \cong G$ π_1 looks good

More discs and relations

$\Gamma_{\mathbb{Z}/3\mathbb{Z}} =$

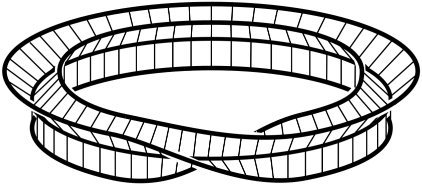


$\longrightarrow \mathbb{Z}/3\mathbb{Z} \circlearrowright \tilde{X}_{\mathbb{Z}/3\mathbb{Z}} \simeq$



- The CW complex $\tilde{X}_G$ is constructed by gluing discs for each $r \in R$ Discs
- G acts on $\tilde{X}_G$, so we obtain $X_G = \tilde{X}_G / G$ Relations

$X_{\mathbb{Z}/3\mathbb{Z}} \simeq \Bigg($



plus glued disc along the boundary circle $\Bigg)$

- Covering $\tilde{X}_G \to X_G$ and $\pi_1(\tilde{X}_G) \cong 1$ gives $\pi_1(X_G) \cong G$ π_1 looks good

For completeness: A formal definition/statement

Given a group G by generators–relations, *i.e.* $G \cong \langle S \mid R \rangle$, the Cayley complex $\tilde{X}_G$ of G is defined by:

(a) $\tilde{X}_G$ is 2-dimensional CW complex

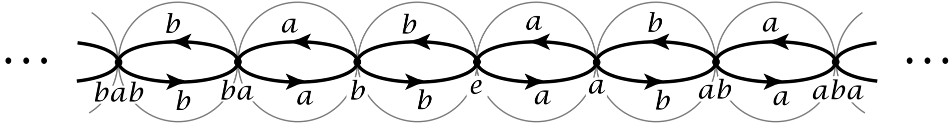
(b) The 0-cell and 1-cells form the Cayley graph Γ_G

(c) For each $g \in G$ and $r \in R$ there is a 2-cell $e_{g,r}$

(d) $e_{g,r}$ is glued to Γ_G starting at g and reading along r Discs

- $\tilde{X}_G$ has a free action of G
- $\pi_1(\tilde{X}_G) \cong 1$
- This gives a covering $\tilde{X}_G \to X_G = \tilde{X}_G / G$ Relations
- $\pi_1(X_G) \cong G$ π_1 looks good

Infinitely many examples



- For $G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \langle a, b \mid a^2 = b^2 = 1 \rangle$ the Cayley complex $\tilde{X}_{\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}}$ is a chain of spheres
- Elements of $\langle ab \rangle \subset G$ act by translation on $\tilde{X}_{\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}}$
- Other elements act as antipodal maps on one sphere and flip the rest end-for-end
- The quotient $X_{\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}}$ is $\mathbb{R}P^2 \vee \mathbb{R}P^2$
- $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

Any group can be realized via a Cayley complex

Thank you for your attention!

I hope that was of some help.