

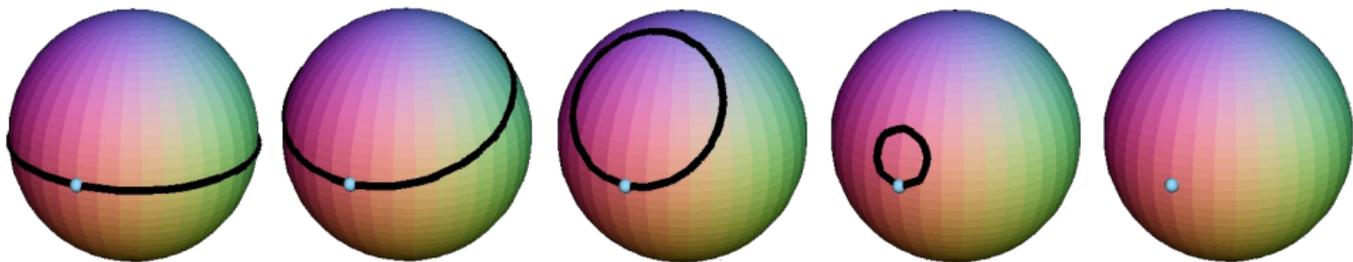
What are...examples of fundamental groups?

Or: One of my favorite lists

$\pi_1(S^1) \cong \mathbb{Z}$ and otherwise:

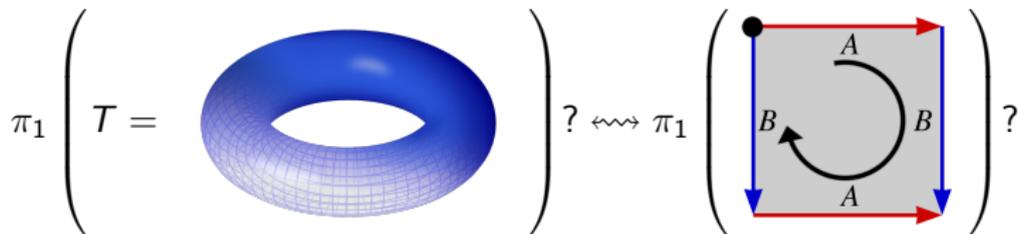
$$\pi_1 \left(S^2 = \text{soccer ball} \right) ?$$

► One directly (pushing curves from the north pole) gets $\pi_1(S^2) \cong 1$:

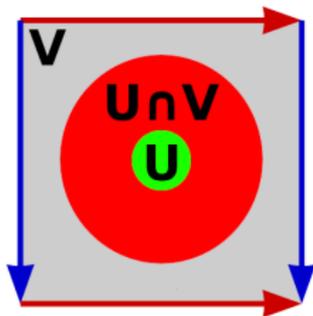


► The same method works for any S^n as long as $n > 1$

Surfaces and polygons



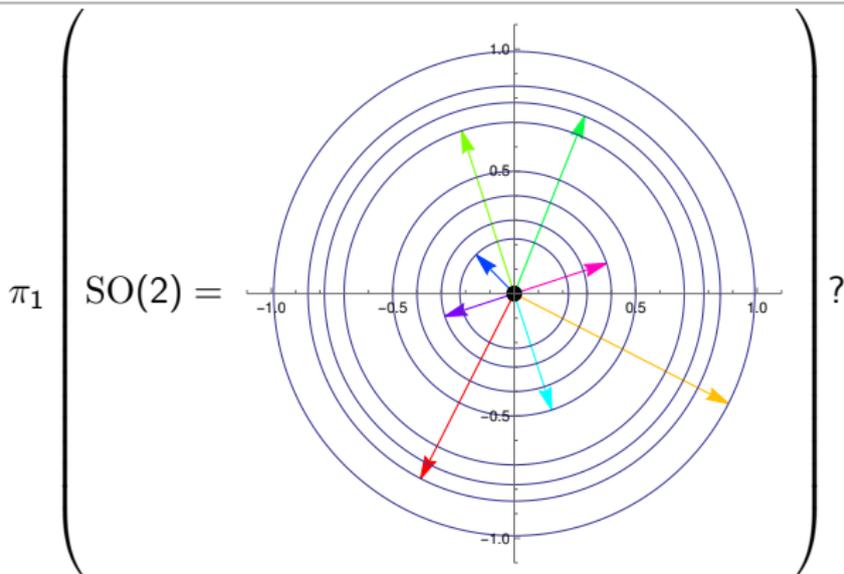
- Using Seifert–van Kampen one gets $\pi_1(T) \cong \langle A, B \mid ABA^{-1}B^{-1} \rangle \cong \mathbb{Z}^2$:



$$\begin{aligned} \pi_1(U) &\cong 1 \\ \pi_1(U \cap V) &\cong \langle ABA^{-1}B^{-1} \rangle \cong \mathbb{Z} \\ \pi_1(V) &\cong \langle A, B \rangle \end{aligned}$$

- The same method works for any surface given by its fundamental polygon

Topological groups



- ▶ $\pi_1(\text{SO}(2)) \cong \mathbb{Z}$ is commutative and loop concatenation can be described via

$$“(\gamma \circ \gamma')(x) \sim \gamma(x) \cdot \gamma'(x)”$$

where \cdot is matrix multiplication

- ▶ The same method works for any topological group

For completeness: A list

Here is a list of important fundamental groups

- ▶ Spheres S^n

$$\pi_1(S^n) \cong \begin{cases} 1 & \text{if } n > 1 \\ \mathbb{Z} & \text{if } n = 1 \end{cases}$$

- ▶ Torus T , real projective plane $\mathbb{R}P^2$ and Klein bottle K

$$\pi_1(T) \cong \mathbb{Z}^2, \quad \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}, \quad \pi_1(K) \cong \langle A_1, B_1, A_2, B_2 \mid A_1 B_1 A_1 B_1^{-1} \rangle$$

- ▶ Orientable surfaces $M_{g,b}$ of genus $g > 0$ and b boundary points

$$\pi_1(M_{g,b}) \cong \langle A_1, B_1, \dots, A_g, B_g, z_1, \dots, z_b \mid [A_1, B_1] \cdot [A_g, B_g] = z_1 \dots z_b \rangle$$

- ▶ Various topological groups G/\mathbb{C} (all have commutative fundamental group)

G	\mathbb{R}	\mathbb{Q}	$GL(n)$	$SL(n)$	$SO(2)$	$SO(> 2)$	$Sp(n)$
π_1	1	1	\mathbb{Z}	1	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$	1

- ▶ Fundamental group of a graph Γ is $\pi_1(\Gamma) \cong *_{e} \mathbb{Z}$ where e runs over edges not contained in a spanning tree (discussed in a previous video)

Genus and boundary

- Without boundary it is the same argument as before

$$\pi_1 \left(M_{2,0} = \left(\text{Diagram 1} = \text{Diagram 2} \right) \right) \cong \langle A, B, C, D \mid ABA^{-1}B^{-1}CDC^{-1}D^{-1} \rangle$$

The diagram shows the fundamental group of a genus-2 surface $M_{2,0}$. On the left, a torus is shown with two handles, each with a loop labeled A and B . The boundary of each handle is labeled C and D . On the right, the same surface is represented as a polygon with boundary segments labeled A, B, C, D in a specific sequence, with a central loop representing the commutator of the boundary segments.

- We also know π_1 of the b -times punctured disc D_b

$$\pi_1 \left(D_b = \left(\text{Diagram} \right) \right) \cong \langle z_1, \dots, z_b \rangle$$

The diagram shows a disc D_b with b punctures labeled z_1, z_2, \dots, z_b . Arrows point from each puncture to a central point, representing the generators of the fundamental group.

- Compute $\pi_1(M_{g,1})$ + glue the boundary of $M_{g,1}$ with the outer boundary D_{b+1}

Thank you for your attention!

I hope that was of some help.