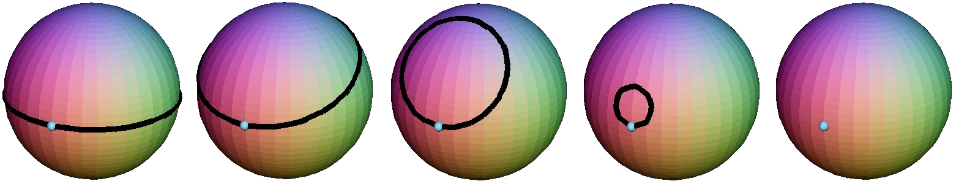
What are...examples of fundamental groups?

Or: One of my favorite lists

$\pi_1(S^1) \cong \mathbb{Z}$ and otherwise:

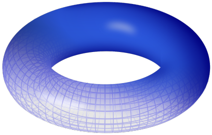
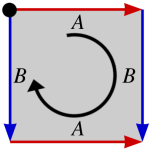
$\pi_1 \Big(S^2 =$  $\Big)$?

- One directly (pushing curves from the north pole) gets $\pi_1(S^2) \cong 1$:



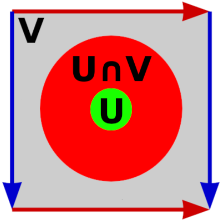
- The same method works for any S^n as long as $n > 1$

Surfaces and polygons

$\pi_1\Big(T =$  $\Big)$? $\leftrightsquigarrow$ $\pi_1\Big($ 

$\Big)$?

- Using Seifert–van Kampen one gets $\pi_1(T) \cong \langle A, B \mid ABA^{-1}B^{-1} \rangle \cong \mathbb{Z}^2$:

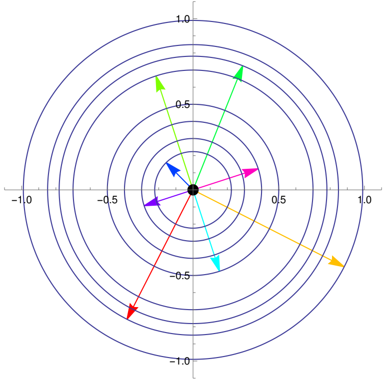


$$\pi_1(U) \cong 1$$
$$\pi_1(U \cap V) \cong \langle ABA^{-1}B^{-1} \rangle \cong \mathbb{Z}$$
$$\pi_1(V) \cong \langle A, B \rangle$$

- The same method works for any surface given by its fundamental polygon

Topological groups

$$\pi_1\left(\mathrm{SO}(2) = \right.$$



$$\left.\right) ?$$

- $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$ is commutative and loop concatenation can be described via

$$\text{“}(\gamma \circ \gamma')(x) \sim \gamma(x) \cdot \gamma'(x)\text{”}$$

  where $\cdot$ is matrix multiplication
- The same method works for any topological group

For completeness: A list

Here is a list of important fundamental groups

- Spheres S^n

$$\pi_1(S^n) \cong \begin{cases} 1 & \text{if } n > 1 \\ \mathbb{Z} & \text{if } n = 1 \end{cases}$$

- Torus T, real projective plane $\mathbb{R}P^2$ and Klein bottle K

$$\pi_1(T) \cong \mathbb{Z}^2, \quad \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}, \quad \pi_1(K) \cong \langle A_1, B_1, A_2, B_2 \mid A_1 B_1 A_1 B_1^{-1} \rangle$$

- Orientable surfaces $M_{g,b}$ of genus $g > 0$ and b boundary points

$$\pi_1(M_{g,b}) \cong \langle A_1, B_1, ..., A_g, B_g, z_1, ..., z_b \mid [A_1, B_1] \cdot [A_g, B_g] = z_1 ... z_b \rangle$$

- Various topological groups $G/\mathbb{C}$ (all have commutative fundamental group)

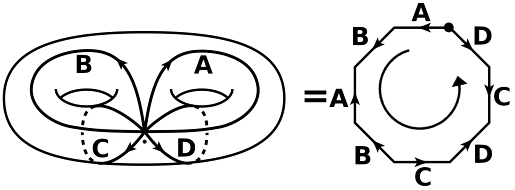
G	$\mathbb{R}$	$\mathbb{Q}$	$\mathrm{GL}(n)$	$\mathrm{SL}(n)$	$\mathrm{SO}(2)$	$\mathrm{SO}(>2)$	$\mathrm{Sp}(n)$
π_1	1	1	$\mathbb{Z}$	1	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1

- Fundamental group of a graph Γ is $\pi_1(\Gamma) \cong *_e \mathbb{Z}$ where e runs over edges not contained in a spanning tree (discussed in a previous video)

Genus and boundary

- Without boundary it is the same argument as before

$\pi_1 \Bigg(M_{2,0} =$

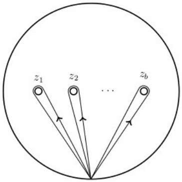


$\Bigg)$

$$\cong \langle A, B, C, D \mid ABA^{-1}B^{-1}CDC^{-1}D^{-1} \rangle$$

- We also know π_1 of the b-times punctured disc D_b

$\pi_1 \Bigg(D_b =$



$\Bigg) \cong \langle z_1, \dots, z_b \rangle$

- Compute $\pi_1(M_{g,1})$ + glue the boundary of $M_{g,1}$ with the outer boundary D_{b+1}

Thank you for your attention!

I hope that was of some help.