What is...the characteristic polynomial?

Or: Matrix roots.

Annihilating polynomials.

$$M = \begin{pmatrix} 1 & 5 & 8 & 10 \\ 0 & 2 & 6 & 9 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \qquad p(X) = (X-1)(X-2)(X-3)(X-4)$$

$$= X^4 - 10X^3 + 35X^2 - 50X + 24.$$

$$p(M) = \begin{pmatrix} 0 & 5 & 8 & 10 \\ 0 & 1 & 6 & 9 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 5 & 8 & 10 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 5 & 8 & 10 \\ 0 & -1 & 6 & 9 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 5 & 8 & 10 \\ 0 & -2 & 6 & 9 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

What on earth is happening?

How can we understand this miracle?

Bad idea. Calculate M^4 *etc.*:

$$M^4 = \begin{pmatrix} 1 & 15 & 210 & 1285 \\ 0 & 16 & 130 & 930 \\ 0 & 0 & 81 & 525 \\ 0 & 0 & 0 & 256 \end{pmatrix}, \dots$$

Better idea. Look at the zeros:

$$\begin{pmatrix} -2 & 5 & 8 & 10 \\ 0 & -1 & 6 & 9 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 5 & 8 & 10 \\ 0 & -2 & 6 & 9 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -4 & -10 & 5 \\ 0 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Can you see how this continues when multiplying with the factors for $(X-1)$ $(X-2)$?

The characteristic polynomial $p(X)$ of a matrix M in general?

Here is another expression that annihilates M, namely

$$p(X) = \det(Xid - M).$$

(Beware: $p(M) = \det(Mid - M) = 0$ is bogus, but works ;-))

Upshot: We can calculate $p(X)$ without knowing the eigenvalues of M.

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad X id - M = \begin{pmatrix} X - 1 & -2 \\ -3 & X - 3 \end{pmatrix}$$

$$p(X) = \det(Xid - M) = (X - 1)(X - 4) - 6.$$

For completeness: A formal definition.

The characteristic polynomial $p(X)$ of a matrix M is defined as $p(X) = \det(Xid - M)$. This polynomial

- ...annihilates M, *i.e.* $p(M) = 0$.
- ...is a product $p(X) = (X - \lambda_1)...(X - \lambda_n)$ for λ_i being the eigenvalues of M (this needs an algebraically closed field).

A cool application.

Try to find M^{100} for some matrix M. For example:

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, p(X) = X^2 - 5X - 2.$$

We have

$$p(M) = 0 \Rightarrow M^2 = 5M + 2.$$

Now calculate further

$$Mp(M) = 0 \Rightarrow M^3 - 5M^2 - 2M = 0 \Rightarrow M^3 = 5(5M + 2) + 2M = 27M + 10.$$

$$M^3 = 27 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}.$$

It follows *verbatim* that all powers of M are linear combinations of M and id.

Thank you for your attention!

I hope that was of some help.