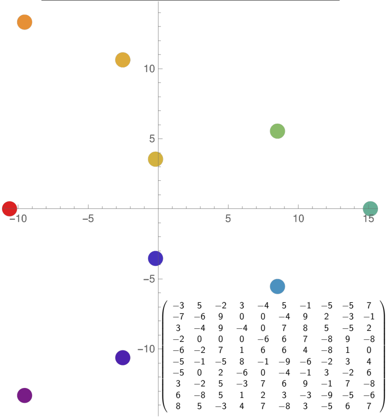
What is...the spectral theorem?

Or: My eigenvalues are real!

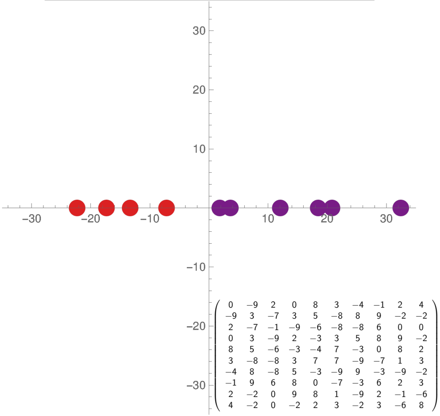
A real matrix has complex eigenvalues, but...

A matrix has complex eigenvalues



$$\begin{pmatrix} -3 & 5 & -2 & 3 & -4 & 5 & -1 & -5 & -5 & 7 \\ -7 & -6 & 9 & 0 & 0 & -4 & 9 & 2 & -3 & -1 \\ 3 & -4 & 9 & -4 & 0 & 7 & 8 & 5 & -5 & 2 \\ -2 & 0 & 0 & 0 & -6 & 6 & 7 & -8 & 9 & -8 \\ -6 & -2 & 7 & 1 & 6 & 6 & 4 & -8 & 1 & 0 \\ -5 & -1 & -5 & 8 & -1 & -9 & -6 & -2 & 3 & 4 \\ -5 & 0 & 2 & -6 & 0 & -4 & -1 & 3 & -2 & 6 \\ 3 & -2 & 5 & -3 & 7 & 6 & 9 & -1 & 7 & -8 \\ 6 & -8 & 5 & 1 & 2 & 3 & -3 & -9 & -5 & -6 \\ 8 & 5 & -3 & 4 & 7 & -8 & 3 & -5 & 6 & 7 \end{pmatrix}$$

A symmetric matrix – real eigenvalues?

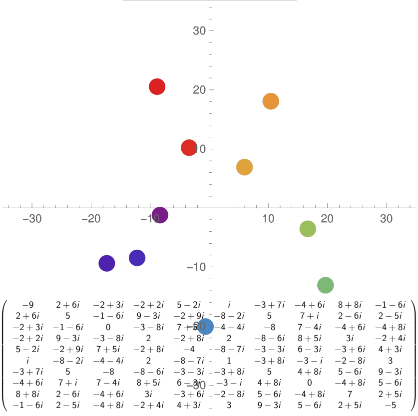


$$\begin{pmatrix} 0 & -9 & 2 & 0 & 8 & 3 & -4 & -1 & 2 & 4 \\ -9 & 3 & -7 & 3 & 5 & -8 & 8 & 9 & -2 & -2 \\ 2 & -7 & -1 & -9 & -6 & -8 & -8 & 6 & 0 & 0 \\ 0 & 3 & -9 & 2 & -3 & 3 & 5 & 8 & 9 & -2 \\ 8 & 5 & -6 & -3 & -4 & 7 & -3 & 0 & 8 & 2 \\ 3 & -8 & -8 & 3 & 7 & 7 & -9 & -7 & 1 & 3 \\ -4 & 8 & -8 & 5 & -3 & -9 & 9 & -3 & -9 & -2 \\ -1 & 9 & 6 & 8 & 0 & -7 & -3 & 6 & 2 & 3 \\ 2 & -2 & 0 & 9 & 8 & 1 & -9 & 2 & -1 & -6 \\ 4 & -2 & 0 & -2 & 2 & 3 & -2 & 3 & -6 & 8 \end{pmatrix}$$

The difference? The right matrix is real symmetric $M = M^T$

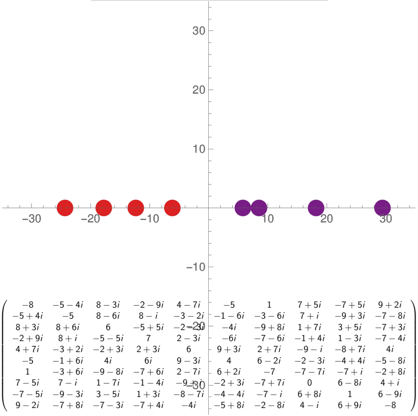
And the complex case?

Looks pretty crazy...



$$\begin{pmatrix} -9 & 2+6i & -2+3i & -2+2i & 5-2i & i & -3+7i & -4+6i & 8+8i & -1-6i \\ 2+6i & 5 & -1-6i & 9-3i & -2+9i & -8-2i & 5 & 7+i & 2-6i & 2-5i \\ -2+3i & -1-6i & 0 & -3-8i & 7+5i & -4-4i & -8 & 7-4i & -4+6i & -4+8i \\ -2+2i & 9-3i & -3-8i & 2 & -2+8i & 2 & -8-6i & 8+5i & 3i & -2+4i \\ 5-2i & -2+9i & 7+5i & -2+8i & -4 & -8-7i & -3-3i & 6-3i & -3+6i & 4+3i \\ i & -8-2i & -4-4i & 2 & -8-7i & 1 & -3+8i & -3-i & -2-8i & 3 \\ -3+7i & 5 & -8 & -8-6i & -3-3i & -3+8i & 5 & 4+8i & 5-6i & 9-3i \\ -4+6i & 7+i & 7-4i & 8+5i & 6-3i & -3-i & 4+8i & 0 & -4+8i & 5-6i \\ 8+8i & 2-6i & -4+6i & 3i & -3+6i & -2-8i & 5-6i & -4+8i & 7 & 2+5i \\ -1-6i & 2-5i & -4+8i & -2+4i & 4+3i & 3 & 9-3i & 5-6i & 2+5i & -5 \end{pmatrix}$$

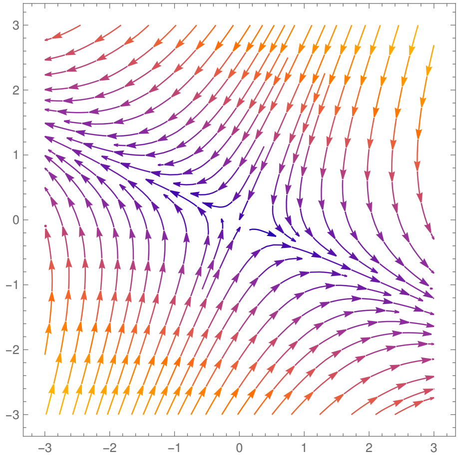
Only real eigenvalues again



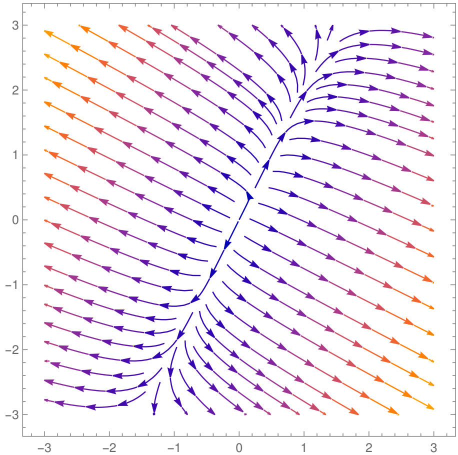
$$\begin{pmatrix} -8 & -5-4i & 8-3i & -2-9i & 4-7i & -5 & 1 & 7+5i & -7+5i & 9+2i \\ -5+4i & -5 & 8-6i & 8-i & -3-2i & -1-6i & -3-6i & 7+i & -9+3i & -7-8i \\ 8+3i & 8+6i & 6 & -5+5i & -2-3i & -4i & -9+8i & 1+7i & 3+5i & -7+3i \\ -2+9i & 8+i & -5-5i & 7 & 2-3i & -6i & -7-6i & -1+4i & 1-3i & -7-4i \\ 4+7i & -3+2i & -2+3i & 2+3i & 6 & 9+3i & 2+7i & -9-i & -8+7i & 4i \\ -5 & -1+6i & 4i & 6i & 9-3i & 4 & 6-2i & -2-3i & -4+4i & -5-8i \\ 1 & -3+6i & -9-8i & -7+6i & 2-7i & 6+2i & -7 & -7-7i & -7+i & -2+8i \\ 7-5i & 7-i & 1-7i & -1-4i & -9+i & -2+3i & -7+7i & 0 & 6-8i & 4+i \\ -7-5i & -9-3i & 3-5i & 1+3i & -8-7i & -4-4i & -7-i & 6+8i & 1 & 6-9i \\ 9-2i & -7+8i & -7-3i & -7+4i & -4i & -5+8i & -2-8i & 4-i & 6+9i & -8 \end{pmatrix}$$

The difference? The right is its own conjugate transpose $M = \overline{M^T}$

Wow, the eigenvectors are orthogonal



$$\begin{pmatrix} 2 & -6 \\ -6 & -9 \end{pmatrix}$$



$$\begin{pmatrix} 9 & -4 \\ -4 & 3 \end{pmatrix}$$

For completeness: A formal statement

If $M\colon V \to V$ is an Hermitian operator, then M has only real eigenvalues and there exists an orthonormal basis of V consisting of eigenvectors of A

Thus, M is diagonalizable with real eigenvalues

Important facts:

(a) For $V = \mathbb{R}^n$ with standard inner product Hermitian means $M = M^T$

Symmetric

(b) For $V = \mathbb{C}^n$ with standard inner product Hermitian means $M = \overline{M^T}$

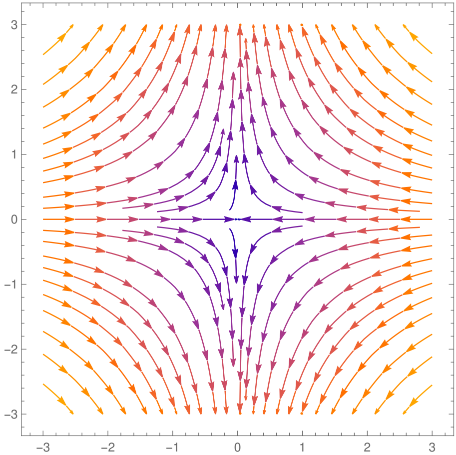
Conjugate transpose

(c) A variant also works for infinite-dimensional vector spaces

Compact self-adjoint operators

Spectral changes happen continuously

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$



Here for
$a = -10$
$b = 0$
$c = 10$

- For $b = 0$ the matrix is diagonal
- The orthogonal system changes continuously

Thank you for your attention!

I hope that was of some help.