

What is...the dual vector spaces?

Or: Two flipped sides.

My wish list for duality (in mathematics and beyond).

V is some object, V^* its dual. The duality $_*$ should satisfy:

- ▶ I want $(V^*)^* \cong V$
- ▶ I want that an operation $V \rightarrow W$ turns into an operation $W^* \rightarrow V^*$
- ▶ I want that operations $V \rightarrow W^*$ correspond to operations $W^* \rightarrow V$

Duality in mathematics is not a theorem, but a “principle” (Atiyah)

Everything that is true about V has a dual statement which is true about V^*

Two points meet in one line, two lines meet in one point:



Strictly speaking this needs projective geometry...

Let us look at linear maps

Consider the usual inner product on \mathbb{R}^3 :

$$\langle v, w \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Another way of writing this:

$$(v_1 \quad v_2 \quad v_3) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$\langle v, - \rangle: \mathbb{R}^3 \rightarrow \mathbb{R}, \quad w \mapsto \langle v, w \rangle \iff (\langle v, (1,0,0) \rangle \quad \langle v, (0,1,0) \rangle \quad \langle v, (0,0,1) \rangle) = (v_1 \quad v_2 \quad v_3)$$

Each vector $v \in \mathbb{R}^3$ corresponds to a linear map $\langle v, - \rangle: \mathbb{R}^3 \rightarrow \mathbb{R}$ and *vice versa*

The transpose space

\mathbb{R}^2 as column vectors

\mathbb{R}^2 as row vectors

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{R}^2 \mid a, b \in \mathbb{R} \right\}$$

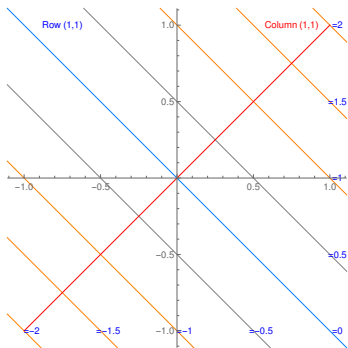
$$(\mathbb{R}^2)^* = \left\{ (a \ b) : \mathbb{R}^2 \rightarrow \mathbb{R} \mid a, b \in \mathbb{R} \right\}$$

The difference?

$\begin{pmatrix} a \\ b \end{pmatrix}$ takes a **number** λ and gives a **vector** $\lambda \begin{pmatrix} a \\ b \end{pmatrix}$

$(a \ b)$ takes a **vector** $\begin{pmatrix} c \\ d \end{pmatrix}$ and gives a **number** $ac + bc$

Example ($a = b = 1$). One of them is one-dimensional, the other kind of as well



For completeness: A formal definition.

The dual vector space is $V^* = \text{End}_K(V, \mathbb{K})$

Important facts for V finite-dimensional be careful with infinities:

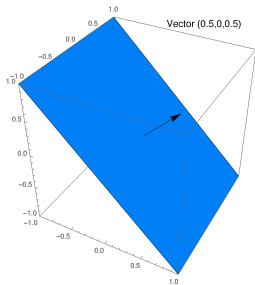
- ▶ Let $B = \{v_1, \dots, v_n\}$ be a basis of vector space V . The dual vector space $V^* = \text{End}_K(V, \mathbb{K})$ has basis $B = \{v_1^*, \dots, v_n^*\}$ given by linear maps

$$v_i^* : V \rightarrow \mathbb{K}, \quad v_i^*(v_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

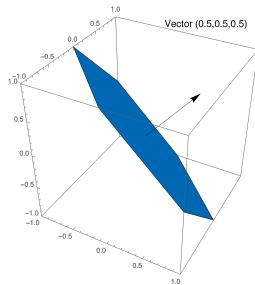
- ▶ $(V^*)^* \cong V$ canonically by $v^{**}(\phi) = \phi(v)$
- ▶ $f : V \rightarrow W$ gives $f^* : W^* \rightarrow V^*$ by $f(\phi) = \phi \circ f$
- ▶ $f : V \rightarrow W^*$ is the same as $f^* : W \rightarrow V^*$ by $f(v) = w^* \Leftrightarrow f^*(w) = v^*$

Dimensions turn around

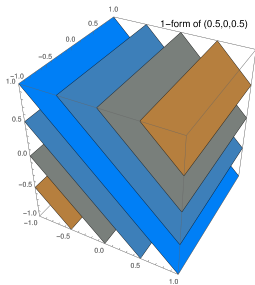
$$v = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}$$



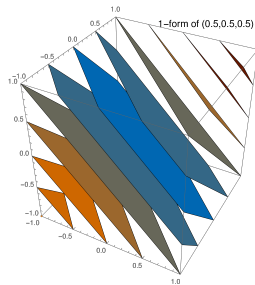
$$v = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}$$



$$\langle v, - \rangle$$



$$\langle v, - \rangle$$



Thank you for your attention!

I hope that was of some help.